Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. ???? (??)** : 1–6.
ISSN-0037-8712 in press
doi:10.5269/bspm.63347

A Generalization of a Result on Generating Functions of Modified Laguerre Polynomial by Using the Notion of Partial Quasibilinear Generating Function

Amartya Chongdar and Prakash Mukherjee

ABSTRACT: In his paper [2], Chongdar obtained an extension (Theorem 3) of the result on bilateral generating functions involving modified Laguerre polynomial stated in Theorem 1 of Ghosh [3].

In this article, the present authors have made an attempt to present a further generalization of the extension obtained by Chongdar [2] by means of the theory of one parameter group of continuous transformations as well as using the concepts of partial quasibilateral generating function [4] involving some special functions.

Key Words: Modified Laguerre polynomial, quasibilateral (or quasibilinear) generating function, quasibilateral (or quasibilinear) generating relation, partial quasibilateral (or Partial quasibilinear) generating function, partial quasibilateral (or Partial quasibilinear) generating relation.

Contents

1. Preliminary concepts and introduction

Special functions are the solutions of a wide class of mathematically and physically relevant functional equations. Generating functions play a large role in the study of special functions. The generating functions which are available in the literature are almost bilateral in nature. There is dearth of trilateral generating functions. Apart from these, some other terms viz quasibilateral (or quasibilinear), partial quasibilateral (or partial quasibilinear), partial semibilateral (or partial semibilinear) generating function etc. are also found in the literature.

In the present paper, we have considered a problem involving partial quasibilinear generating function [4] which is defined by the following generating relation:

$$G(x,z,w)=\sum_{n=0}^{\infty}a_n w^n P_{m+n}^{(\alpha)}(x)Q_r^{(m+n)}(z), \tag{1.1}$$

where a_n are arbitrary constants and independent of x, z. $P_{m+n}^{(\alpha)}(x)$ and $Q_r^{(m+n)}(z)$ are two special functions of orders $m+n, r$ and of parameters $\alpha, m+n$ respectively. If, in particular, the above two special functions are of same in nature i.e. $Q_r^{(m+n)}(z) \equiv P_r^{(m+n)}(z)$, we call the generating relation (1.1) as partial quasi-bilnear generating relation. For example, if we replace $P_{m+n}^{(\alpha)}(x)$, $Q_r^{(m+n)}(z)$ by $L_{m+n}^{(\alpha)}(x)$, $C_r^{(m+n)}(z)$ respectively, then (1.1) is called partial quasibilateral generating relation involving Laguerre and Gegenbauer polynomials [5] and if we replace $P_{m+n}^{(\alpha)}(x)$, $Q_r^{(m+n)}(z)$ by the same type of polynomials, $L_{m+n}^{(\alpha)}(x)$, $L_r^{(m+n)}(z)$ (say), we call the relation (1.1) as partial quasibilinear generating relation involving Laguerre polynomials [6] etc.

2010 *Mathematics Subject Classification:* 33C45.
Submitted April 25, 2022. Published September 26, 2022

There are different methods of obtaining generating functions for various special functions. In this problem, we have used the group-theoretic method which has been receiving much attention in recent years. The idea of group-theoretic method in the investigation of generating functions for various special functions was introduced by L. Weisner [7,8,9] while investigating Hypergeometric, Hermite and Bessel functions.

In [3], Ghosh obtained the following result on bilateral generating function involving modified Laguerre polynomial.

Theorem 1.1. *If*

$$G(x,t) = \sum_{n=0}^{\infty} a_n f_n^{\beta-n}(x)\ t^n, \tag{1.2}$$

then

$$\sum_{n=0}^{\infty} f_n^{\beta-n}(x)\sigma_n(y)t^n = (1+t)^{\beta-1} exp\Big(\frac{xt}{1+t}\Big) G\Big(\frac{x}{1+t}, \frac{yt}{1+t}\Big) \tag{1.3}$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p \binom{n}{p} y^p.$$

Subsequently, an extension of the above theorem was found derived in [2] in the following form :

Theorem 1.2. *If there exists a generating relation of the form,*

$$G(x,t) = \sum_{n=0}^{\infty} a_n f_{n+k}^{\beta-n}(x) t^n, \tag{1.4}$$

then

$$\sum_{n=0}^{\infty} f_{n+k}^{\beta-n}(x)\sigma_n(y)t^n = (1+t)^{\beta-1} exp\Big(\frac{xt}{1+t}\Big) G\Big(\frac{x}{1+t}, \frac{yt}{1+t}\Big) \tag{1.5}$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p \frac{(k+p+1)_{n-p}}{(n-p)!} y^p.$$

The above theorems are very much important for their usefulness in the generalization of known results.

The object of the present paper is to further generalize the Theorem 1.2 by means of group-theoretic method based on the theory of one parameter group of continuous transformations and by using the concept of partial quasibilateral generating function as defined in [4]. The main result of this paper is given in the form of the following theorem.

Theorem 1.3. *If there exists a generating relation of the form :*

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n f_{n+r}^{\beta-n}(x) f_m^{n+r}(u) w^n, \tag{1.6}$$

then

$$(1+w)^{-m-r}(1-w)^{\beta-1}exp\Big(\frac{-xw}{1-w}\Big)G\Big(\frac{x}{1-w}, u(1+w), \frac{wv}{(1-w)(1+w)}\Big)$$
$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} v^n(-1)^{p+q}(n+r+1)_p(n+r)_q f_{n+r+p}^{\beta-n-p}(x) f_m^{n+r+q}(u). \tag{1.7}$$

The above theorem did not seem to appear before. In the next section we shall prove it.

2. Proof of the theorem 1.3

To prove the theorem 1.3, we first consider two linear partial differential operators R_1 and R_2, each of which generates one parameter continuous transformations group, as follows :

$$R_1 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} - y(\beta + x - 1)$$

,

$$R_2 = ut\frac{\partial}{\partial u} - t^2\frac{\partial}{\partial t} - (m+r)t$$

such that

$$R_1\Big(f_{n+r}^{\beta-n}(x)y^n\Big) = -(n+r+1)f_{n+r+1}^{\beta-n-1}(x)y^{n+1} \tag{2.1}$$

$$R_2\Big(f_m^{n+r}(u)t^n\Big) = -(n+r)f_m^{n+r+1}(u)t^{n+1} \tag{2.2}$$

and

$$e^{wR_1}f(x,y) = exp\Big(-\frac{xyw}{1-yw}\Big)(1-yw)^{\beta-1}f\Big(\frac{x}{1-yw}, \frac{y}{1-yw}\Big) \tag{2.3}$$

$$e^{wR_2}f(u,t) = (1+wt)^{-m-r}f\Big(u(1+wt), \frac{t}{1+wt}\Big). \tag{2.4}$$

Let us now consider the generating relation :

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n f_{n+r}^{\beta-n}(x) f_m^{n+r}(u) w^n. \tag{2.5}$$

Replacing w by $wytv$ on both sides of (2.5), we get

$$G(x,u,wytv) = \sum_{n=0}^{\infty} a_n \Big(f_{n+r}^{\beta-n}(x)y^n\Big)\Big(f_m^{n+r}(u)t^n\Big)(wv)^n. \tag{2.6}$$

Now operating $e^{wR_1}e^{wR_2}$ on both sides of (2.6), we get

$$e^{wR_1}e^{wR_2}G(x,u,wytv) = e^{wR_1}e^{wR_2}\Bigg[\sum_{n=0}^{\infty} a_n \Big(f_{n+r}^{\beta-n}(x)y^n\Big)\Big(f_m^{n+r}(u)t^n\Big)(wv)^n\Bigg]. \tag{2.7}$$

The left member of (2.7), with the help of (2.3) and (2.4), becomes

$$(1+wt)^{-m-r}exp\Big(-\frac{xyw}{1-yw}\Big)(1-yw)^{\beta-1}G\Big(\frac{x}{1-yw}, u(1+wt), \frac{wytv}{(1-yw)(1+wt)}\Big). \tag{2.8}$$

The right member of (2.7), with the help of (2.1) and (2.2), becomes

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} v^n (-1)^{p+q} (n+r+1)_p (n+r)_q f_{n+r+p}^{\beta-n-p}(x) f_m^{n+r+q}(u)\ y^{n+p}\ t^{n+q}. \tag{2.9}$$

Now equating (2.8) and (2.9) and finally putting $y = t = 1$, we get

$$(1+w)^{-m-r}(1-w)^{\beta-1} exp\Big(\frac{-xw}{1-w}\Big) G\Big(\frac{x}{1-w}, u(1+w), \frac{wv}{(1-w)(1+w)}\Big)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} v^n (-1)^{p+q} (n+r+1)_p (n+r)_q f_{n+r+p}^{\beta-n-p}(x) f_m^{n+r+q}(u),$$

which is relation (1.7). This completes the proof of the theorem.

Corollary 2.1. *Putting $r = 0$ in theorem 1.3, we get the following:*
If there exists a quasibilinear generating relation [1]:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n f_n^{\beta-n}(x) f_m^n(u) w^n$$

then

$$(1+w)^{-m}(1-w)^{\beta-1} exp\Big(\frac{-xw}{1-w}\Big) G\Big(\frac{x}{1-w}, u(1+w), \frac{wv}{(1-w)(1+w)}\Big)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} v^n (-1)^{p+q} (n+1)_p (n)_q f_{n+p}^{\beta-n-p}(x) f_m^{n+q}(u),$$

which shows that the existence of a quasibilinear generating relation involving modified Laguerre polynomial implies the existence of a more general generating relation.

We now proceed to show that theorem 1.3 is a generalization of theorem 1.2 by discussing the particular case of theorem 1.3 when $m = 0$.

3. Particular Case

Putting m=0 in (1.7), we get

$$(1+w)^{-r}(1-w)^{\beta-1} exp\Big(\frac{-xw}{1-w}\Big) G\Big(\frac{x}{1-w}, \frac{wv}{(1-w)(1+w)}\Big)$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{w^{n+p}}{p!} v^n (-1)^p (n+r+1)_p f_{n+r+p}^{\beta-n-p}(x) \sum_{q=0}^{\infty} \frac{(-w)^q}{q!} (n+r)_q$$
$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{w^{n+p}}{p!} v^n (-1)^p (n+r+1)_p f_{n+r+p}^{\beta-n-p}(x) (1+w)^{-n-r}$$
$$= (1+w)^{-r} \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n (-w)^{n+p} \Big(\frac{-v}{1+w}\Big)^n \frac{(n+r+1)_p}{p!} f_{n+r+p}^{\beta-n-p}(x)$$
$$= (1+w)^{-r} \sum_{n=0}^{\infty} (-w)^{n+p} \sum_{p=0}^{\infty} a_n \frac{(n+r+1)_p}{p!} \Big(\frac{-v}{1+w}\Big)^n f_{n+r+p}^{\beta-n-p}(x) \tag{3.1}$$

Now replacing $\big(-\frac{v}{1+w}\big)$ by v in (3.1), we get

$$(1-w)^{\beta-1} exp\Big(\frac{-xw}{1-w}\Big) G\Big(\frac{x}{1-w}, \frac{-wv}{1-w}\Big) = \sum_{n=0}^{\infty} (-w)^n \sum_{p=0}^{n} a_p \binom{n+r}{p+r} v^p f_{n+r}^{\beta-n}(x). \tag{3.2}$$

Finally, writing (w) in place $(-w)$ in (3.2), we get

$$(1+w)^{\beta-1} exp\Big(\frac{xw}{1+w}\Big) G\Big(\frac{x}{1+w}, \frac{wv}{1+w}\Big) = \sum_{n=0}^{\infty} w^n \sigma_n(v) f_{n+r}^{\beta-n}(x),$$

where

$$\sigma_n(v) = \sum_{p=0}^{n} a_p \binom{n+r}{p+r} v^p$$

and

$$G(x,w) = \sum_{n=0}^{\infty} a_n f_{n+k}^{\beta-n}(x) w^n,$$

which is Theorem 1.2.
Thus we see that Theorem 1.3 is a further generalization of Theorem 1.2.

4. Observation

It is observed that though the Theorem 1.3 has been proved by group theoretic method, still the result stated in Theorem 1.3 owes its existence to the following generating functions :

$$exp\Big(\frac{xt}{1+t}\Big)(1+t)^{\beta-1-n} f_{n+r}^{\beta-n}\Big(\frac{x}{1+t}\Big) = \sum_{p=0}^{\infty} \frac{(n+r+1)_p}{p!} f_{n+r+p}^{\beta-n-p}(x)\ t^p, \tag{4.1}$$

$$(1-t)^{-m-r-n} f_m^{n+r}\big(u(1-t)\big) = \sum_{q=0}^{\infty} \frac{(n+r)_q}{q!} f_m^{(n+r+q)}(u)\ t^q \tag{4.2}$$

as well as to the partial quasibilinear generating function assumed in Theorem 1.3.

In fact R. H. S of (1.7)

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} v^n (-1)^{p+q} (n+r+1)_p (n+r)_q f_{n+r+p}^{\beta-n-p}(x) f_m^{n+r+q}(u)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{(-w)^p}{p!} (n+r+1)_p f_{n+r+p}^{\beta-n-p}(x) \left(\sum_{q=0}^{\infty} \frac{(n+r)_q}{q!} f_m^{(n+r+q)}(u)\ (-w)^q\right) (wv)^n$$

$$=(1+w)^{-m-r} \sum_{n=0}^{\infty} a_n \left(\sum_{p=0}^{\infty} \frac{(n+r+1)_p}{p!} f_{n+r+p}^{\beta-n-p}(x)\ (-w)^p\right) f_m^{n+r}\big(u(1+w)\big) \left(\frac{vw}{1+w}\right)^n \text{ (using (4.2))}$$

$$=(1+w)^{-m-r}(1-w)^{\beta-1} exp\Big(\frac{-xw}{1-w}\Big) \sum_{n=0}^{\infty} a_n\ f_{n+r}^{\beta-n}\Big(\frac{x}{1-w}\Big) f_m^{n+r}\big(u(1+w)\big) \left(\frac{vw}{(1+w)(1-w)}\right)^n \text{ (using (4.1)}$$

$$=(1+w)^{-m-r}(1-w)^{\beta-1} exp\Big(\frac{-xw}{1-w}\Big) G\Big(\frac{x}{1-w}, u(1+w), \frac{wv}{(1-w)(1+w)}\Big)$$

$$=L.H.S\ of\ (1.7)$$

Acknowledgments

The authors are grateful to Professor A. K. Chongdar of the Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, for being instrumental in suggesting the problem. Also we would like to thank the referee for some valuable suggestions towards the improvement of the paper.

References

1. Chatterjea S. K. and Chakraborty S. P., *A unified group-theoretic method of obtaing a more general class of generating relations from a given class of quasi bilateral (or quasi bilinear) generating relations involving some special functions*, Pure Math. Manuscript 8, 153–162, (1989).

2. Chongdar, A. K., *On a class of bilateral generating functions for certain special functions*, Proceedings of the Indian Academy of Sciences-Mathematical Sciences 95, 133-140, (1986).

3. Ghosh, B., *Some theorems on generating functions of modified Laguerre Polynomials*, Bull. Cal. Math. Soc. 79, 41-47, (1987).

4. Mondal, A. K., *A unified group-theoretic method of obtaining a more general class of generating relations from a given class of partial quasi-bilateral (or partial quasi-bilinear) generating relations involving some special functions*, Pure Math. Manuscript 10, 121-132, (1992-93).

5. Mondal, A. K., *Partial quasi-bilateral generating function involving Laguerre and Gegenbauer polynomials (I)*, Pure Math. Manuscript 10, 97-103, (1992-93).

6. Mondal, A. K., *Partial quasi-bilinear generating function involving Laguerre polynomials*, Pure Math. Manuscript 10, 89-95, (1993).

7. Weisner, L., *Group - theoretic origin of certain generating functions*, Pacific J. Math. 5, 1033–1039, (1955).

8. Weisner, L., *Generating functions for Hermite functions*, Canad. J. Math. 11, 141–147, (1959).

9. Weisner, L., *Generating functions for Bessel functions*, Canad. J. Math. 11, 148–155, (1959).

Amartya Chongdar,
Department of Mathematics,
Bangabasi College,
India.
E-mail address: acmath77@gmail.com

and

Prakash Mukherjee,
Department of Mathematics,
Hijli College,
India.
E-mail address: prakashmukherjee25@gmail.com