Some Fixed Point Theorems in Generalized $M$-Fuzzy Metric Space

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ABSTRACT: In this paper, we define the expansive mapping in $G$-metric space and we prove some fixed point theorems in generalized $M$-fuzzy ($GM$-fuzzy) metric space.

Key Words: Fuzzy metric space, $G$-metric space, $GM$-fuzzy metric space, Expansive mapping.

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1. Introduction

The theory of fuzzy sets was introduced by Zadeh [18]. Thereafter the introduced notion has evolved in many directions of science and technology, where mathematics has a role. It has been studied by Tripathy and Borgohain [11], Tripathy and Dutta [12] for studying the properties of sequences of fuzzy numbers, Tripathy and Ray [16] for studying fuzzy topological spaces, Deb and Saha [1], Dhange [2], Mustafa et al [5], Sedghi et al [9], Sun and Yang [10], Tripathy et al ([13], [14], [15]), Wang [17] and others for studying fixed point theory in fuzzy settings. Different researcher have interpreted and introduced the concept of fuzzy metric space in different ways. George and Veeramani [3] modified the concept of a fuzzy metric space introduced by Kramosil and Michalek [4] and defined a Hausdorff topology on this fuzzy metric space.

The study of fixed points of a function satisfying certain contractive conditions has been at the center of rigorous research activity. Mustafa and Sims [7] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa et al [8] obtained some fixed point theorems for mappings satisfying different contractive conditions.

2. Preliminaries and Definitions

Definition 2.1: A fuzzy set $M$ on an arbitrary set $X$ is a function with domain $X$ and range in $[0, 1]$.

Definition 2.2: A binary operation $*$ : $[0, 1] 	imes [0, 1] \to [0, 1]$ is called a continuous $t$-norm if $(0, 1, *)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 \leq a_2 * b_2$ whenever $a_1 \leq a_2, b_1 \leq b_2$ for all $a_1, a_2, b_1, b_2 \in [0, 1]$.

Examples of $t$-norm

1) Minimum $t$-norm ($*M$) : $*M(x, y) = \min\{x, y\}$.
2) Product $t$-norm ($*P$) : $*P(x, y) = x \cdot y$.
3) Lukasiewicz $t$-norm ($*L$) : $*L(x, y) = \max\{x + y - 1, 0\}$.

Definition 2.3: Let $X$ be a non-empty set and let $G : X \times X \times X \to R^+$, be a function satisfying the following properties:

$(G_1)G(x, x, y) > 0$, for all $x, y \in X$, with $x \neq y$;
$(G_2)G(x, y, z) = 0$, if $x = y = z$.

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\[ (G_3) G(x, x, y) \leq G(x, y, z) \text{ for all } x, y, z \in X \text{ with } z = y; \]
\[ (G_4) G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots; \]
(Symmetry in all three variables)
\[ (G_5) : G(x, y, z) \leq G(x, a, a) + G(a, y, z) \text{ for all } x, y, z, a \in X; \]
(rectangle inequality).

Then the function \( G \) is called Generalized metric or more specifically \( G \)-metric on \( X \), and the pair \((X, G)\) is called a \( G \)-metric space.

**Definition 2.4** The 3-tuple \((X, M, \ast)\) is called a fuzzy metric if \( X \) is an arbitrary set, \( \ast \) is a continuous \( t \)-norm and \( M \) is a fuzzy set in \( X^2 \times (0, \infty) \) satisfying the following conditions, for all \( x, y, z \in X \) and \( t_1, t_2, t > 0 \),
\[ (1) M(x, y, 0) = 0; \]
\[ (2) M(x, y, t) = 1 \text{ if and only if } x = y; \]
\[ (3) M(x, y, t) = M(y, x, t); \]
\[ (4) M(x, y, t_1 + t_2) \geq M(x, z, t_1) \ast M(z, y, t_2); \]
\[ (5) M(x, y, \cdot) : (0, \infty) \to [0, 1] \text{ is continuous.} \]

Then \( M \) is called fuzzy metric on \( X \) and \((X, M, \ast)\) is called fuzzy metric space and \( M(x, y, t) \) denotes the degree of nearness between \( x \) and \( y \).

**Definition 2.5** A 3-tuple \((X, M, \ast)\) is said to be a Generalized \( M(GM) \)-fuzzy metric space if \( X \) is an arbitrary non-empty set, \( \ast \) is a continuous \( t \)-norm and \( M \) is a fuzzy set on \( X^3 \times (0, \infty) \) satisfying the following conditions for each \( t, s > 0 \):
\[ (M1) M(x, x, y, t) > 0 \text{ for all } x, y \in X \text{ with } x \neq y; \]
\[ (M2) M(x, x, y, t) \geq M(x, y, z, t) \text{ for all } x, y, z \in X \text{ with } y \neq z; \]
\[ (M3) M(x, y, z, t) = 1 \text{ if and only if } x = y = z; \]
\[ (M5) M(x, a, a, t) \ast M(a, y, z, s) \leq M(x, y, z, t + s); \] (the triangle inequality)
\[ (M6) M(x, y, \cdot) : (0, \infty) \to [0, 1] \text{ is continuous.} \]

A \( GM \)-fuzzy metric space is said to be symmetric if \( M(x, y, t) = M(x, y, t) \) for all \( x, y \in X \) and \( t > 0 \).

**Example 2.1** Let \( X \) be a non-empty set and \( G \) be the \( G \)-metric on \( X \).
Denote \( a \ast b = ab \) for all \( a, b \in [0, 1] \), For each \( t > 0 \):
\[ M(x, y, z, t) = \frac{t}{t + G(x, y, z)}. \]

Then \((X, M, \ast)\) is a \( GM \)-fuzzy metric space.

**Definition 2.6** Let \((X, M, \ast)\) be a \( GM \)-fuzzy metric space. Then
\[ (a) \text{ A sequence } \{x_n\} \text{ in } X \text{ is said to coverage to } x \text{ if and only if } M(x_m, x_n, x, t) \to 1 \text{ as } n \to \infty, m \to \infty, \text{ for each } t > 0. \]
\[ (b) \text{ A sequence } \{x_n\} \text{ in } X \text{ is said to be a } G \text{-Cauchy sequence if } M(x_m, x_n, x, t) \to 1 \text{ as } m \to \infty, n \to \infty, t \to \infty \text{ for each } t > 0. \]
\[ (c) \text{ A } GM \text{-fuzzy metric space in which every Cauchy sequence is convergent is said to be } G \text{-complete.} \]

**Lemma 2.7** If \((X, M, \ast)\) be a \( GM \)-fuzzy metric space, then \( M(x, y, z, t) \) is non-decreasing with respect to \( t \) for all \( x, y, z \in X \).

Throughout this article we assume that \( \lim_{n \to \infty} M(x_n, y, z, t) = 1 \) and that \( N \) is the set of all natural numbers and that \( R^+ \) is the set of all positive real numbers.

**Lemma 2.8.** Let \((X, M, \ast)\) be a \( GM \)-fuzzy metric space. Then the following properties are equivalent:
1) \( \{x_n\} \) is convergent to \( x \).
2) \( M(x_n, x_n, x, t) \to 1, \) as \( n \to \infty \).
3) \( M(x_n, x, x, t) \to 1, \) as \( n \to \infty \).
4) \( M(x_m, x_n, x, t) \to 1, \) as \( m, n \to \infty \).

Lemma 2.9. Let \((X, M, \ast)\) be a GM-fuzzy metric space, then the following are equivalent:

1) The sequence \( \{x_n\} \) is G-Cauchy.
2) For every \( \varepsilon \in (0, 1) \) and \( t > 0 \), there exists \( k \in N \) such that
\[
M(x_n, x_m, x, t) > 1 - \varepsilon \quad \text{for} \quad n, m \geq k.
\]

Definition 2.10. Let \((X, M, \ast)\) be a GM-fuzzy metric space. The following conditions are satisfied:

\[
\lim_{n \to \infty} M(x_n, y_n, z_n, t_n) = M(x, y, z, t),
\]

whenever \( \lim_{n \to \infty} x_n = x, \lim_{n \to \infty} y_n = y, \lim_{n \to \infty} z_n = z \)
and \( \lim_{n \to \infty} M(x, y, z, t_n) = M(x, y, z, t) \),

then \( M \) is called a continuous function on \( X^3 \times (0, \infty) \).

Lemma 2.11 Let \((X, M, \ast)\) be a GM-fuzzy metric space. Then \( M \) is a continuous function on \( X^3 \times (0, \infty) \).

Lemma 2.12 Let \((X, M, \ast)\) be a complete GM-fuzzy metric space and \( T : X \to X \) be a mapping satisfies the following conditions for all \( x, y, z \in X \) and \( t > 0 \),

\[
kM(Tx, Ty, Tz, t) \geq M(x, y, z, t), \quad \text{where} \quad k \in [0, 1)
\]

Lemma 2.13 Let \((X, M, \ast)\) be a complete GM-fuzzy metric space and \( T : X \to X \) be a mapping satisfies the following conditions for all \( x, y \in X \) and \( t > 0 \)

\[
kM(Tx, Ty, Ty, t) \geq M(x, y, y, t),
\]

where \( k \in [0, 1) \). Then \( T \) has a unique fixed point.

Definition 2.14 Let \((X, M, \ast)\) be a GM-fuzzy metric space and \( T \) be a self mapping on \( X \). Then \( T \) is called expansive mapping if there exists a constant \( a \geq 1 \), such that for all \( x, y, z \in X \) and \( t > 0 \), we have

\[
M(Tx, Ty, Tz, t) \geq aM(x, y, z, t).
\]

3. Main Results

Theorem 3.1 Let \((X, M, \ast)\) be a complete GM-fuzzy metric space. If there exists a constant \( a \leq 1 \) and a onto self mapping \( T \) on \( X \), such that for all \( x, y, z \in X \) and \( t > 0 \), we have

\[
M(Tx, Ty, Tz, t) \leq aM(x, y, z, t).
\]

Then \( T \) has a unique fixed point.

Proof. Under the assumption, if \( Tx = Ty \), then

\[
1 = M(Tx, Ty, Ty, t) \leq aM(x, y, y, t).
\]

Which implies \( M(x, y, y, t) = 1 \Rightarrow x = y. \)

Hence, \( T \) is injective and invertible.

Let \( h \) be the inverse mapping of \( T \), then \( M(x, y, z, t) = M(T(hx), T(hy), T(hz), t) \leq aM(hx, hy, hz, t). \)

Thus, for all \( x, y, z \in X \) and \( t > 0 \).
we have, \( aM(hx, hy, hz, t) \geq M(x, y, z, t) \).

Applying Lemma 2.12, we conclude that inverse mapping \( h \) has a unique fixed point \( u \in X; h(u) = u \). But, \( u = T(h(u)) = T(u) \). This gives that \( u \) is also a fixed point of \( T \).

Suppose there exists another fixed point \( v \neq u \) such that \( Tv = v \).
Then, \( Tv = v = T(h(v)) = h(T(v)) \).

So, \( Tv \) is another fixed point of \( h \).

By uniqueness, we conclude that \( u = Tv = v \), which implies that \( u \) is a unique fixed point of \( T \).

**Theorem 3.2** Let \((X, M, \ast)\) be a complete GM-fuzzy metric space. If there exists a constant \( c \leq 1 \) and a surjective self mapping \( T \) on \( X \), such that for all \( x, y \in X \) and \( t > 0 \),
\[
M(Tx, Ty, Ty, t) \leq cM(x, y, y, t),
\]
Then \( T \) has a unique fixed point.

**Proof.** Under the assumption, if \( Tx = Ty \), then
\[
1 = M(Tx, Tx, Ty, t) \leq cM(x, x, y, t)
\]
Which implies \( M(x, x, y, t) = 1 \).

\( \Rightarrow x = y \)
and hence \( T \) is invertible.

Let \( h \) be the inverse mapping of \( T \),
So, \( M(x, y, y, t) = M(T(hx), T(hy), T(hy), t) \leq cM(hx, hy, hy, t) \).
Then, for all \( x, y \in X \), we have
\[
cM(hx, hy, hy, t) \geq M(x, y, y, t).
\]
Applying Lemma 2.12 on the inverse mapping \( h \), and use argument similar to that in Proof Theorem 3.1, we conclude that \( T \) has unique fixed point.

**Corollary 3.3.** Let \((X, M, \ast)\) be a complete GM-fuzzy metric space. If there exists a constant \( k \leq 1 \) and surjective self mapping on \( X \), such that for all \( x, y, z \in X \) and \( t > 0 \),
\[
M(Tx, Ty, Tz, t) \leq k \max\left\{ (M(hx, hy, hy, t/2) \ast M(hy, hy, hy, t/2)), (M(hy, hy, hy, t/2)) \right\},
\]
Then \( T \) has a unique fixed point.

**Proof.** The proof follows from Theorem 3.2 by taking \( z = y \) in condition (3.2).

**Theorem 3.4** Let \((X, M, \ast)\) be a complete GM-fuzzy metric space and let \( T : X \rightarrow X \) be a surjective mapping satisfying the following condition for all \( x, y, z \in X \) and \( t > 0 \),
\[
M(T(x), T(y), T(z), t) \leq k \max\left\{ (M(x, z, z, t/2) \ast M(y, z, z, t/2)), (M(z, y, y, t/2)) \ast M(x, x, t/2)) \right\},
\]
Then \( T \) has a unique fixed point.

**Proof.** Condition (3.3) implies \( T \) is injective and therefore invertible.
Let \( h \) be the inverse mapping of \( T \).
By condition (4), for all \( x, y, z \in X, t > 0 \) We have,
\[
M(x, y, z, t) = M(T(hx), T(hy), T(hz), t) \leq k \max\left\{ (M(hx, hy, hy, t/2) \ast M(hy, hy, hy, t/2)), (M(hy, hy, hy, t/2)) \right\}
\]
where \( k \leq 1 \). Then \( T \) has a unique fixed point.
By \((M4)\), we have

\[
\max\{ (M(hx, hz, hz, t/2) * M(hy, hy, hy, t/2) ), (M(hz, hy, hy, t/2) * M(hx, hy, hy, t/2)) \} \leq M(hx, hy, hy, t).
\]

Thus equation \((3.4)\) implies

\[
kM(hx, hy, hy, t) \geq M(x, y, z, t).
\]

Applying, Theorem 3.1 with the help of \((3.6)\).

We conclude that the inverse mapping \(h\) has a unique fixed point \(u \in X\) such that \(h(u) = u\).

But \(u = T(h(u)) = T(u)\),

Which shows that \(u\) is also a fixed point of \(T\).

To show \(u\) is unique fixed point, we can use the same argument in Theorem 3.4.

**Theorem 3.5:** Let \((X, M, *)\) be a complete non-symmetric \(GM\)-fuzzy metric space and let \(T : X \to X\) be a surjective mapping satisfying the following condition for all \(x, y, z, t > 0\),

\[
M(T(x), T(y), T(z), t) \leq k\max\{ M(x, y, y, t), M(y, x, x, t) \}. \tag{3.7}
\]

When \(k \leq 1\). Then \(T\) has a unique fixed point.

**Proof:** Since \(\max\{ M(x, y, y, t), M(y, x, x, t) \} \leq M(x, y, y, t)\), then from \((3.7)\), we deduce

\[
M(T(x), T(y), T(z), t) \leq kM(x, y, y, t) \text{ for all } x, y \in X, t > 0. \tag{3.8}
\]

From \((3.8)\), it is clear that Theorem 3.2 implies that \(T\) has a unique fixed point.

**Corollary 3.6:** Let \((X, M, *)\) be a complete non-symmetric \(GM\)-fuzzy metric space, and let \(T : X \to X\) be a surjective mapping satisfying the following condition for all \(x, y, z, t > 0\),  

\[
M(T(x), T(y), T(z), t) \leq k\max\{ (M(x, y, y, t/2) * M(y, x, x, t/2)) , (M(z, z, t/2) * M(z, z, t/2)) \} \tag{3.5}
\]

when \(k \leq 1\). Then \(T\) has a unique fixed point.

**Proof:** Follows from the Theorem 3.5 on taking \(z = y\).

**Corollary 3.7:** Let \((X, M, *)\) be a complete \(GM\)-fuzzy metric space and let \(T : X \to X\) be a surjective mapping satisfying the following condition for all \(x, y, z \in X, t > 0\),

\[
M(T(x), T(y), T(z), t) \leq k\{ M(x, Tz, Tz, t/2) * M(Tx, y, z, t/2) \}, \tag{3.9}
\]

where \(k \leq 1\). Then \(T\) has a unique fixed point.

**Proof:** From \((M4)\), we have

\[
M(x, Tx, Tx, t/2) * M(Tx, y, z, t/2) \leq M(x, y, z, t).
\]

Then condition \((10)\) becomes

\[
M(T(x), T(y), T(z), t) \leq kM(x, y, z, t) \text{ for all } x, y, z \in X \text{ and the proof follows from } (3.1).
\]

**Theorem 3.8:** Let \((X, M, *)\) be a complete \(GM\)-fuzzy metric space and \(T : X \to X\) be an onto and continuous mapping satisfying the following conditions for all \(x \in X\) and \(t > 0\),

\[
M(T(x), T^2(x), T^3(x), t) \leq aM(x, T(x), T^2x, t). \tag{3.10}
\]

Where \(a \leq 1\). Then \(T\) has a fixed point.
Proof: : Let $x_0 \in X$, since $T$ is onto, so there exists an element $x_1$ satisfying $x_1 \in T^{-1}(x_0)$. By the same argument we can pick up $x_n \in T^{-1}(x_{n-1})$ where $n = 2, 3, 4, 5, \ldots$. Let $x_n \neq x_{n-1}$, then there is a sequence $x_n$ with $x_n \neq x_{n-1}$ and $T(x_n) = x_{n-1}$.

Then (3.10) implies
\[ M(x_{n-1}, x_{n-2}, x_{n-3}, t) = M(Tx_n, T^2x_n, T^3x_n, t) \leq aM(x_n, TTx_n, T^2Tx_n) \]
\[ = aM(x_n, x_{n-1}, x_{n-2}, t). \]  
Therefore, we have
\[ M(x_n, x_{n-1}, x_{n-2}, t) \geq \frac{1}{a} M(x_{n-1}, x_{n-2}, x_{n-3}, t). \]

Let $q = \frac{1}{a}$, then $q \geq 1$.

It can be easily verified that the sequence $\{ x_n \}$ is a Cauchy and by completeness of $(X, M, \ast)$, the sequence $\{ x_n \}$ converges to a point $u \in X$.

Since $T$ is continuous, then $T(x_n) = x_{n-1} \rightarrow T(u)$ as $n \rightarrow \infty$.

Hence, $T(u) = u$, which shows that $u$ is a fixed point of $T$.

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