The Continuous Generalized Wavelet Transform Associated with $q$-Bessel Operator

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ABSTRACT: The continuous generalized wavelet transform associated with $q$-Bessel operator is defined, which will invariably be called continuous $q$-Bessel wavelet transform. Certain and boundedness results and inversion formula for continuous $q$-Bessel wavelet transform are obtained. Discrete $q$-Bessel wavelet transform is defined and a reconstruction formula is derived for discrete $q$-Bessel wavelet.

Key Words: $q$-Bessel Function, $q$-Bessel Fourier transform, Wavelet transform.

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1. Introduction

A complex-valued continuous function $\phi$ with the property

$$\int_{0}^{\infty} \phi(t) dt = 0,$$

is called a wavelet. The wavelet transform of a function $f \in L^2(\mathbb{R})$ with respect to the wavelet $\phi \in L^2(\mathbb{R})$ is defined by

$$(W_\phi)(b, a) = \int_{-\infty}^{+\infty} f(t) \overline{\phi_{b,a}(t)} dt, \ b \in \mathbb{R}, \ a > 0,$$

where

$$\phi_{b,a}(t) = a^{-1/2} \phi((t - b)/a).$$

In terms of the translation $T_b$ defined by

$$T_b \phi(t) = \phi(t - b), \ b \in \mathbb{R}$$

and dilation $D_a$ defined by

$$D_a \phi(t) = |a|^{-1/2} \phi(t/a), \ a \neq 0,$$

we can write

$$\phi_{b,a}(t) = T_b D_a \phi(t).$$
We can also express (1.2) as the convolution:

\[(W_\phi f)(b, a) = (f * g_{\phi, a})(b),\]  

where

\[g(t) := \phi(-t).\]  

(1.8)

2. The \(q\)-Bessel operator and \(q\)-Bessel function

The \(q\)-Bessel operator defined by

\[\Delta_{q,a} f (x) = \frac{1}{x^{2\alpha+1}} D_q \left[ x^{2\alpha+1} D_q f \right] \left( q^{-1} x \right), \]  

where

\[D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \] \(x \neq 0, q \neq 1.\)  

(2.2)

For \(a, q \in \mathbb{C},\) the \(q\)-shift factorial \((a; q)_k\) is defined as a product of \(k\) factors

\[(a; q)_k = (1 - a)(1 - aq) \ldots (1 - aq^{k-1}), k \in \mathbb{N}^*, (a; q)_0 = 1.\]  

(2.3)

If \(|q| < 1,\) this definition remains meaningful for \(k = +\infty\) as a convergent infinite product:

\[(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k).\]  

(2.4)

We also write \((a_1, \ldots, a_r; q)_k\) for the product of \(rq\)-shifted factorials:

\[(a_1, \ldots, a_r; q)_k = (a_1; q)_k \ldots (a_r; q)_k, k \in \mathbb{N} \text{ or } k = \infty.\]  

(2.5)

A \(q\)-hypergeometric series is a power series (for the moment still formal) in one complex variable \(z\) with power series coefficients which depend, apart from \(q,\) on \(r\) complex upper parameters \(a_1, \ldots, a_r\) and \(s\) complex lower parameters \(b_1, \ldots, b_s\) as follows:

\[r_{\phi, s}(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \frac{(-1)^k q^{k(k-1)}}{(1 - q^2)^{1+s-r}} x^k, \text{ for } r, s \in \mathbb{N}.\]  

(2.6)

The \(q\)-Bessel function is defined by

\[j_\alpha (x; q^2) = \Gamma_{q^2} (\alpha + 1) \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)}}{\Gamma_{q^2} (k + 1) \Gamma_{q^2} (\alpha + k + 1)} \left( \frac{x}{1 + q} \right)^{2k}.\]  

(2.7)

This function is bounded and for every \(x \in \mathbb{R}_q\) and \(\alpha > -\frac{1}{2},\) we have

\[|j_\alpha (x; q^2)| \leq \frac{1}{(q; q^2)^2},\]  

(2.8)

\[\left( \frac{1}{x} D_q \right) j_\alpha (x; q^2) = -\frac{(1 - q)}{(1 - q^{2\alpha + 2})} j_{\alpha - 1} (qx; q^2),\]  

(2.9)

\[\left( \frac{1}{x} D_q \right) \left( x^{2\alpha} j_\alpha (x; q^2) \right) = \frac{(1 - q^{2\alpha})}{(1 - q)} x^{2(\alpha - 1)} j_{\alpha - 1} (x; q^2),\]  

(2.10)

\[|D_q j_\alpha (x; q^2)| \leq \frac{x (1 - q)}{(1 - q^{2\alpha + 2}) (q; q^2)^2}.\]  

(2.11)
We remark that for $\lambda \in C$, the function $j_\alpha (\lambda x, q^2)$ is the unique solution of the $q$-differential system

$$\begin{cases} \Delta_{q,\alpha} U (x, q) = - \lambda^2 U (x, q) \\ U (0, q) = 1 ; D_{q,x} U (x, q) |_{x=0} = 0, \end{cases}$$

(2.12)

where $\Delta_{q,\alpha}$ is the $q$-Bessel operator defined by

$$\Delta_{q,\alpha} f (x) = \frac{1}{x^{2\alpha + 1}} D_{q} \left[ x^{2\alpha + 1} D_{q} f \right] (q^{-1} x)$$

(2.13)

and for $k \in N$ and $\lambda \in R_q^+$

$$\Delta_{q,x}^k j_\alpha (\lambda x; q^2) = (-1)^k \lambda^{2k} j_\alpha (\lambda x; q^2).$$

(2.16)

3. $q$-Functional spaces

We begin by putting

$$R_q^+ = \{ + q^k, k \in Z \}, \quad \tilde R_q^+ = \{ + q^k, k \in Z \} \cup \{0\}$$

(3.1)

and we denote by $L_{p,\alpha,q} (R_{q,+})$, $p \leq [0, \infty[$, (resp. $L_{\infty,\alpha,q} (R_{q,+})$) the space of functions $f$ such that,

$$\|f\|_{p,\alpha,q} = \left( \int_0^\infty |f (x)|^p \, dq(x) \right)^{\frac{1}{p}} < +\infty,$$

(3.2)

$$\text{resp.} \quad \|f\|_{\infty,\alpha,q} = \text{ess. sup}_{x \in R_q} |f (x)| < +\infty,$$

(3.3)

and

$$dq(x) = \frac{(1 + q)^{-\alpha}}{\Gamma q^2 (\alpha + 1)} x^{2\alpha + 1} dqx = b_{\alpha,q} x^{2\alpha + 1} dqx.$$

(3.4)

4. $q$-Bessel translation operator

$T_{q,x}^\alpha$, $x \in R_{q,+}$ is the $q$-generalized translation operator associated with the $q$-Bessel transform is introduced in [12], is defined as follows

$$\phi (x, y) = T_{q,x}^\alpha f (x) = \int_0^{+\infty} f (t) D_{\alpha,q} (x, y, t) \, dq\sigma (t), \quad \alpha > -1,$$

(4.1)

with

$$D_{\alpha,q} (x, y, z) = \int_0^{+\infty} j_\alpha (xt; q^2) j_\alpha (yt; q^2) j_\alpha (zt; q^2) \, dq\sigma (t)$$

(4.2)

and

$$\int_0^{+\infty} D_{\alpha,q} (x, y, z) \, dq\sigma (z) = 1.$$

(4.3)

In particular the following product formula holds

$$T_{q,x}^\alpha j_\alpha (y; q^2) = j_\alpha (x; q^2) j_\alpha (y; q^2).$$

(4.4)

It is shown in [12] that for $f \in L_{p,\alpha,q} (R_{q,+})$

$$\|T_{q,x}^\alpha f\|_{p,\alpha,q} \leq \|f\|_{p,\alpha,q},$$

(4.5)

and the map $y \to T_{q,x}^\alpha f$ is continuous from $(0, \infty)$ into $(0, \infty)$. 


5. \(q\)-Convolution and \(q\)-Bessel Fourier transform

The \(q\)-Bessel Fourier transform \(F_{\alpha,q}\) and the \(q\)-Bessel convolution product are defined for suitable functions \(f, g\) as follows

\[
f_{\alpha,q}(\lambda) = \int_{0}^{\infty} f(x) j_{\alpha}(\lambda x; q^{2}) d_{q}\sigma(x), \quad (5.1)
\]

\[
f \ast_{\alpha,q} g(x) = \int_{0}^{\infty} T_{q,x}^{\alpha} f(y) g(y) d_{q}\sigma(y). \quad (5.2)
\]

It is shown in [11], that the \(q\)-Bessel Fourier transform \(F_{\alpha,q}\) satisfies the following properties:

**Theorem 5.1.** If \(f \in L_{\alpha,q}^{1}(\mathbb{R}_{q,+})\) then \(F_{\alpha,q}(f) \in C_{q,x,0}(\mathbb{R}_{q,+})\) and

\[
\|\hat{f}_{\alpha,q}\| \leq B_{\alpha,q} \|f\|_{1,\alpha,q}, \quad (5.3)
\]

where

\[
B_{\alpha,q} = \frac{1}{(1-q)} \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}}. \quad (5.4)
\]

**Theorem 5.2.** Given two functions \(f, g \in L_{\alpha,q}^{1}(\mathbb{R}_{q,+})\), then

\[
f \ast_{\alpha,q} g \in L_{\alpha,q}^{1}(\mathbb{R}_{q,+}) \quad (5.5)
\]

and

\[
F_{\alpha,q}(f \ast_{\alpha,q} g) = F_{\alpha,q}(f) F_{\alpha,q}(g). \quad (5.6)
\]

**Theorem 5.3.** (Inversion formula): If \(f \in L_{\alpha,q}^{1}(\mathbb{R}_{q,+})\) such that \(F_{\alpha,q}(f) \in L_{\alpha,q}^{1}(\mathbb{R}_{q,+})\), then for all \(x \in \mathbb{R}_{q,+}\), we have

\[
f(x) = \int_{0}^{\infty} \hat{f}_{\alpha,q}(\lambda) j_{\alpha}(xy; q^{2}) d_{q}\sigma(y) \quad (5.7)
\]

**Theorem 5.4.** (\(q\)-Plancherel theorem) If \(\hat{f}_{\alpha,q}\) is an isomorphism of \(L_{\alpha,q}^{2}(\mathbb{R}_{q,+})\), we have

\[
\left\|\hat{f}_{\alpha,q}(\lambda)\right\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}, \text{ for } f \in L_{\alpha,q}^{2}(\mathbb{R}_{q,+}) \text{ and } F_{\alpha,q}^{-1}(f) = F_{\alpha,q}(f). \quad (5.8)
\]

**Theorem 5.5.** (i) For \(f \in L_{\alpha,q}^{p}(\mathbb{R}_{q,+}), p \in [1, \infty]\), \(g \in L_{\alpha,q}^{1}(\mathbb{R}_{q,+})\), we have

\[
f \ast_{\alpha,q} g \in L_{\alpha,q}^{p}(\mathbb{R}_{q,+}) \text{ and } \|f \ast_{\alpha,q} g\|_{p,\alpha,q} \leq \|f\|_{p,\alpha,q} \|g\|_{1,\alpha,q}. \quad (5.9)
\]

(ii) \(\int_{0}^{\infty} F_{\alpha,q}(f)(\xi) g(\xi) d_{q}\sigma(\xi) = \int_{0}^{\infty} f(\xi) F_{\alpha,q}(g)(\xi) d_{q}\sigma(\xi), \quad f, g \in L_{\alpha,q}^{1}(\mathbb{R}_{q,+}). \quad (5.10)
\]

(iii) \(F_{\alpha,q}(T_{q,x}^{\alpha})(\xi) = j_{\alpha}(\xi x; q^{2}) F_{\alpha,q}(f)(\xi), \quad f \in L_{\alpha,q}^{1}(\mathbb{R}_{q,+}). \quad (5.11)
\]

6. The continuous generalized wavelet transform associated with \(q\)-Bessel operator

Let \(\psi \in L_{\alpha,q}^{p}(\mathbb{R}_{q,+}), 1 \leq p < \infty\) be given. For \(b \geq 0 \text{ and } a > 0\) define the \(q\)-Bessel wavelet

\[
\psi_{b,a}^{\alpha,q}(x) := D_{\alpha} B_{\alpha}^{\alpha,q} \psi(x) = D_{\alpha} \psi(b, x) = a^{-2\alpha-2} \psi\left(a^{-1} \frac{b}{a} x\right) \quad (6.1)
\]

\[
a^{-2\alpha-2} \int_{0}^{\infty} D_{\alpha,q} \left(a^{-1} \frac{b}{a} x\right) \psi(z) d_{q}\sigma(z), \quad (6.2)
\]

the integral being convergent by virtue of (4.5).

Using the wavelet \(\psi_{b,a}^{\alpha,q}\), we now define the continuous \(q\)-Bessel wavelet transform which will send each \(L^{p}\)-function defined on the positive half line to a function \(B_{\alpha,q}(b, a)\) on the first quadrant as follows.

\[
B_{\alpha,q}(b, a) := \left(\mathcal{B}^{\alpha,q}_{\psi} f\right)(b, a) := \left\langle f(t), \psi_{b,a}^{\alpha,q}(t) \right\rangle_{\alpha,q} = \int_{0}^{\infty} f(t) \overline{\psi_{b,a}^{\alpha,q}(t)} d_{q}\sigma(t) \quad (6.3)
\]
\( \psi = a^{-2\alpha - 2} \int_0^\infty \int_0^\infty f(t) \psi(z) D_{\alpha,q} \left( \frac{b}{a}, \frac{t}{a}, z \right) d_q \sigma(z) d_q \sigma(t), \) 

provided the integral is convergent; see Theorem 5.3 for existence.

**Theorem 6.1.** Let \( \psi \in L_{p,q}^p (R_{q,+}), \) \( 1 \leq p < \infty. \) Then for \( y \geq 0, \)

(i) the map \( y \rightarrow T_y^{\alpha,q} \psi \) is continuous from \( L_{p,q}^p (R_{q,+}) \) into \( L_{p,q}^p (R_{q,+}). \)

(ii) the function \( \psi_{b,a} \) is defined almost everywhere on \( [0, \infty), \) and

\[
\left\| \psi_{b,a} (x) \right\|_{p,a,q} \leq a^{(2\alpha+2)(\frac{1}{p'} - 1)} \left\| \psi \right\|_{p,a,q}.
\]

**Proof.** We can write, for \( \frac{1}{p} + \frac{1}{p'} = 1, \)

\[
\left| \psi(x,y) \right| = \left| T_y^{\alpha,q} \psi(x) \right| = \left| \int_0^\infty \psi(z) D_{\alpha,q}^{1/p} (x,y,z) D_{\alpha,q}^{1/p'} (x,y,z) \right| d_q \sigma(z)
\]

\[
\leq \left( \int_0^\infty \left| \psi(z) \right|^p D_{\alpha,q}^{1/p} (x,y,z) d_q \sigma(z) \right)^{1/p} \left( \int_0^\infty D_{\alpha,q}^{1/p'} (x,y,z) d_q \sigma(z) \right)^{1/p'}.
\]

Therefore, in view of the property (4.3), we have

\[
\left| \psi(x,y) \right|^p \leq \int_0^\infty \left| \psi(z) \right|^p D_{\alpha,q}^{1/p} (x,y,z) d_q \sigma(z),
\]

so that

\[
\int_0^\infty \left| \psi(x,y) \right|^p d_q \sigma(x) \leq \int_0^\infty \left| \psi(z) \right|^p d_q \sigma(z) \int_0^\infty D_{\alpha,q}^{1/p} (x,y,z) d_q \sigma(x).
\]

Thus, we get the following boundedness property of the \( q \)-Bessel translation operator

\[
\left\| \psi(x,y) \right\|_{p,a,q} \leq \left\| \psi \right\|_{p,a,q}, \quad 1 \leq p < \infty.
\]

(6.6)

Now applying the above method of proof to (6.2) we find that

\[
\left\| \psi_{b,a} (x) \right\|_{p,a,q} \leq a^{(2\alpha+2)(\frac{1}{p'} - 1)} \left\| \psi \right\|_{p,a,q}, \quad 1 \leq p < \infty.
\]

**Theorem 6.2.** Let \( f \in L_{p,q}^p (R_{q,+}) \) and \( \psi \in L_{p,q}^{p'} (R_{q,+}) \) with \( 1 \leq p, p' < \infty \) and \( \frac{1}{p} + \frac{1}{p'} = 1, \) and \( B_{\alpha,q} (b,a) = \left( B_{\psi,q}^{\alpha,q} f \right) (b,a) \) be the continuous \( q \)-Bessel wavelet transform (6.4). Then

(i) \( B_{\alpha,q} (b,a) \) is continuous on \( (0, \infty) \times (0, \infty). \)

(ii) \( \left\| B_{\alpha,q}^{\alpha,q} f \right\|_{r,a,q} \leq a^{(2\alpha+2)/r} \left\| f \right\|_{p,a,q} \left\| \psi \right\|_{p',a,q}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{p'} - 1, \quad 1 \leq p, p', r < \infty.
\]

(iii) \( \left\| B_{\alpha,q}^{\alpha,q} f \right\|_{\infty,a,q} \leq a^{(2\alpha+2)(1/r' - 1)} \left\| f \right\|_{p,a,q} \left\| \psi \right\|_{p',a,q}, \quad \frac{1}{r'} = \frac{1}{p} + \frac{1}{p'} = 1.
\]

**Proof.** (i) Let \( (b_0, a_0) \) be an arbitrary but fixed point in \( (0, \infty) \times (0, \infty). \) Then by Holder's inequality,

\[
\left| B_{\alpha,q} (b,a) - B_{\alpha,q} (b_0,a_0) \right| \leq a^{-2\alpha - 2} \int_0^\infty \int_0^\infty \left| f(t) \psi(z) \right| D_{\alpha,q} (b/a, t/a, z) - D_{\alpha,q} (b_0/a_0, t/a_0, z) d_q \sigma(t) d_q \sigma(z)
\]

\[
\leq a^{-2\alpha - 2} \left( \int_0^\infty \int_0^\infty \left| f(t) \right|^p D_{\alpha,q} (b/a, t/a, z) - D_{\alpha,q} (b_0/a_0, t/a_0, z) d_q \sigma(t) d_q \sigma(z) \right)^{1/p}
\]

\[
\times \left( \int_0^\infty \int_0^\infty \left| \psi(z) \right|^p D_{\alpha,q} (b/a, t/a, z) - D_{\alpha,q} (b_0/a_0, t/a_0, z) d_q \sigma(t) d_q \sigma(z) \right)^{1/p'}.
\]
Therefore, by Holder’s inequality, we have
\[
\left| f(t) \psi(z) D_{\alpha,q} (b/a, t/a, z) d_q \sigma(t) d_q \sigma(z) \right| \leq 2,
\]
by dominated convergence theorem and continuity of \( D_{\alpha,q}(b/a, t/a, z) \) in the variable \( b \) and \( a \), we have
\[
\lim_{b \to b_0, a \to a_0} |B_{\alpha,q}(b, a) - B_{\alpha,q}(b_0, a_0)| = 0.
\]
This prove that \( B_{\alpha,q}(b, a) \) is continuous on \((0, \infty) \times (0, \infty)\).

\[ (iii) \quad (B_{\psi}^{\alpha,q} f)(b, a) = a^{-2\alpha - 2} \int_0^\infty \int_0^\infty f(t) \psi(z) D_{\alpha,q} (b/a, t/a, z) d_q \sigma(t) d_q \sigma(z) \]
\[
= a^{-2\alpha - 2} \int_0^\infty \int_0^\infty f(t) \psi(z) D_{\alpha,q}^{1/p} (b/a, t/a, z) D_{\alpha,q}^{1/p'} (b/a, t/a, z) d_q \sigma(t) d_q \sigma(z).
\]

Therefore, by Holder’s inequality, we have
\[
\left| (B_{\psi}^{\alpha,q} f)(b, a) \right| \leq a^{-2\alpha - 2} \left( \int_0^\infty \int_0^\infty |f(t)|^p D_{\alpha,q} (b/a, t/a, z) d_q \sigma(t) d_q \sigma(z) \right)^{1/p} \times \left( \int_0^\infty \int_0^\infty |\psi(z)|^{p'} D_{\alpha,q} (b/a, t/a, z) d_q \sigma(t) d_q \sigma(z) \right)^{1/p'} \leq a^{-2\alpha - 2} \left( \int_0^\infty |f(t)|^p d_q \sigma(t) \int_0^\infty D_{\alpha,q} (b/a, t/a, z) d_q \sigma(z) \right)^{1/p} \times \left( \int_0^\infty |\psi(z)|^{p'} d_q \sigma(z) \int_0^\infty D_{\alpha,q} (b/a, t/a, z) d_q \sigma(t) \right)^{1/p'} \leq a^{(2\alpha + 2)/(1/p' - 1)} \left( \int_0^\infty |f(t)|^p d_q \sigma(t) \right)^{1/p} \left( \int_0^\infty |\psi(z)|^{p'} d_q \sigma(z) \right)^{1/p'}.
\]
Thus
\[
\left| (B_{\psi}^{\alpha,q} f)(b, a) \right| \leq a^{(2\alpha + 2)/(1/p' - 1)} \|f\|_{p,\alpha,q} \|\psi\|_{p',\alpha,q}.
\]

This proves (iii).
The inequality (ii) follows from Theorem (5.3).

7. An Inversion formula

**Theorem 7.1.** Let \( \psi \in L^2_{\alpha,q}(\mathbb{R}_+,\mathbb{C}) \) be a basic wavelet which defines the continuous \( q \)-Bessel wavelet transform (6.4). Then, for
\[
C_{\alpha,q}^{\psi} = \int_0^\infty \omega^{-2\alpha - 2} |\hat{\psi}(\omega)|^2 d_q \sigma(\omega) > 0,
\]
\[
\int_0^\infty \int_0^\infty (B_{\psi}^{\alpha,q} f)(b, a) (B_{\psi}^{\alpha,q} g)(b, a) a^{-2\alpha - 2} d_q \sigma(a) d_q \sigma(b) = C_{\alpha,q}^{\psi} \langle f, g \rangle_{\alpha,q}, \quad \forall f, g \in L^2_{\alpha,q}(\mathbb{R}_+,\mathbb{C}).
\]
Proof. Using the representation (6.4) we have

\[
\left(B_{\alpha,q}^{\psi} f\right)(b,a) = a^{-2\alpha-2} \int_0^\infty \int_0^\infty f(t) \overline{\psi(z)} D_{\alpha,q} \left( \frac{b \cdot t}{a \cdot z}, \frac{b \cdot \xi}{a \cdot q^2} \right) d_q \sigma(z) d_q \sigma(t)
\]

\[
= a^{-2\alpha-2} \int_0^\infty \int_0^\infty f(t) \overline{\psi(z)} j_\alpha \left( \frac{b \cdot t}{a \cdot q^2} \right) j_\alpha \left( \frac{b \cdot \xi}{a \cdot q^2} \right) d_q \sigma(z) d_q \sigma(t)
\]

\[
= a^{-2\alpha-2} \int_0^\infty \int_0^\infty \hat{f}_{\alpha,q} \left( \frac{t}{a} \right) \overline{\psi_{\alpha,q}(z)} j_\alpha \left( \frac{b \cdot \xi}{a \cdot q^2} \right) d_q \sigma(z) \ d_q \sigma(t)
\]

\[
= \int_0^\infty \hat{f}(\xi) \overline{\psi_{\alpha,q}(a\xi)} \ d_q \sigma(\xi)
\]

Now multiplying by \(a^{-2\alpha-2} d_q \sigma(a)\) and integrating, we get

\[
\int_0^\infty \int_0^\infty \left( B_{\psi}^{\alpha,q} f \right)(b,a) \overline{ \left( B_{\psi}^{\alpha,q} g \right)(b,a) } \ d_q \sigma(b) = \int_0^\infty \int_0^\infty \hat{f}_{\alpha,q}(\xi) \overline{\psi_{\alpha,q}(a\xi)} \ d_q \sigma(\xi) \ \overline{ \hat{g}_{\alpha,q}(\xi) \overline{\psi_{\alpha,q}(a\xi)} } \ d_q \sigma(\xi)
\]

\[
= \int_0^\infty \hat{f}_{\alpha,q}(\xi) \overline{\psi_{\alpha,q}(a\xi)} \ d_q \sigma(\xi) \ int_0^\infty \overline{\psi_{\alpha,q}(a\xi)} \ d_q \sigma(a)
\]

\[
= \int_0^\infty \overline{\hat{f}_{\alpha,q}(\xi) g_{\alpha,q}(\xi)} \ d_q \sigma(\xi) \ \overline{ \int_0^\infty \hat{\psi}_{\alpha,q}(a\xi) \ d_q \sigma(a) } \ d_q \sigma(\xi)
\]

\[
= C_{\alpha,q}^{\psi} \left( f, g \right)_{\alpha,q}
\]

8. Discrete \(q\)-Bessel wavelet transform

In this section we assume that \(\psi \in L^2_{\alpha,q}(\mathbb{R}_{q,+})\) satisfies the so called stability condition

\[
P \leq \sum_{m=-\infty}^{\infty} \left| \hat{\psi}(2^{-m}\xi) \right|^2 \leq Q \text{ a.e.}
\]

(8.1)

for certain positive constants \(P\) and \(Q\), \(0 < P \leq Q < \infty\). Here \(\hat{\psi}\) denotes the \(q\)-Bessel Fourier transform of \(\psi\). The \(\psi \in L^2_{\alpha,q}(\mathbb{R}_{q,+})\) satisfying (8.1) is called dyadic wavelet.

We define the semi-discrete \(q\)-Bessel wavelet transform by

\[
\left( B_{m,q}^{\alpha,q} \psi f \right)(b) := (2^m)^{2\alpha+2} \left( B_{\psi}^{\alpha,q} f \right) \left( b, \frac{1}{2^m} \right)
\]

(8.2)
ψ defined by (8.3). Let us define another wavelet \( \psi \in L^2 M. M. Dixit, C. P. P andey and D. Das \) for the some constants of points only variable \( a \). Now, we discretise the translation parameter \( b \) also by restricting it to the discrete set

\[
\text{Theorem 8.1. Assume that the semi-discrete } q\text{-Bessel wavelet transform of any } f \in L^2_{\alpha, q} (R_+, q) \text{ is defined by (8.3). Let us define another wavelet } \psi^* \text{ by means of its } q\text{-Bessel Fourier transform:}
\]

\[
\hat{\psi}^*_{\alpha, q} (\xi) = \frac{\hat{\psi}_{\alpha, q} (\xi)}{\sum_{k=-\infty}^{\infty} |\hat{\psi}_{\alpha, q} (2^{-k}\xi)|^2}.
\]

then

\[
f (t) = \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} (B^0_{m, q} f (b) \left( \hat{\psi}^*_{\alpha, q} (2^{-m}\xi) j_\alpha (tu; q^2) \right) \hat{\psi}_{\alpha, q} (2^{-m}\eta) j_\alpha (t\xi; q^2)) \hat{\psi}_{\alpha, q} (b) d_\eta d_\sigma (b).
\]

\[
\text{Proof. In view of (8.1) and (8.3), for any } f \in L^2_{\alpha, q} (R_+, q), \text{ we have}
\]

\[
\sum_{m=-\infty}^{\infty} \int_{0}^{\infty} (B^0_{m, q} f (b) \left( \hat{\psi}^*_{\alpha, q} (2^{-m}\xi) j_\alpha (tu; q^2) \right) \hat{\psi}_{\alpha, q} (2^{-m}\eta) j_\alpha (t\xi; q^2)) \hat{\psi}_{\alpha, q} (b) d_\eta d_\sigma (b)
\]

\[
= \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} (B^0_{m, q} f (b) \left( \hat{\psi}^*_{\alpha, q} (2^{-m}\eta) j_\alpha (tu; q^2) \right) \hat{\psi}_{\alpha, q} (2^{-m}\eta) j_\alpha (t\eta; q^2)) \hat{\psi}_{\alpha, q} (b) d_\eta d_\sigma (b)
\]

\[
= \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} \left( \hat{f}_{\alpha, q} (\eta) \left( \hat{\psi}^*_{\alpha, q} (2^{-m}\eta) \right) \hat{\psi}_{\alpha, q} (2^{-m}\eta) j_\alpha (t\eta; q^2) \right) \hat{\psi}_{\alpha, q} (b) d_\eta d_\sigma (b)
\]

\[
= \int_{0}^{\infty} \hat{f}_{\alpha, q} (\eta) j_\alpha (t\eta; q^2) d_\eta d_\sigma (b)
\]

\[
f (t).
\]

The above theorem leads to the following definition of dyadic dual.

\[
\text{Definition 8.2. A function } \tilde{\psi} \in L^2_{\alpha, q} (R_+, q) \text{ is called a dyadic dual of a dyadic wavelet } \psi \text{ if every } f \in L^2_{\alpha, q} (R_+, q) \text{ can be expressed as}
\]

\[
f (t) = \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} (B^0_{m, q} f (b) \left( \hat{\psi} (2^{-m}\xi) j_\alpha (t\xi; q^2) \hat{\psi}_{\alpha, q} (b) \right) \hat{\psi}_{\alpha, q} (2^{-m}\eta) j_\alpha (t\eta; q^2)) \hat{\psi}_{\alpha, q} (b) d_\eta d_\sigma (b).
\]

\[
\text{So far we have considered semi-discrete Bessel wavelet transform of any } f \in L^2_{\alpha, q} (R_+, q) \text{ discretising only variable } a. \text{ Now, we discretise the translation parameter } b \text{ also by restricting it to the discrete set of points}
\]

\[
b_{m,n} := \frac{n}{2^m} b_0, \text{ } m \in Z, \text{ } n \in N_0.
\]
where \( b_0 > 0 \) is a fixed constant.

We write
\[
\psi_{b_0;m,n}^{\alpha,q}(t) = 2^{m(2\alpha+2)} \psi_{\alpha,q}(nb_0,2^mt).
\]

Then the discrete Bessel wavelet transform of any \( f \in L^2_{\alpha,q}(\mathbb{R}^+) \) can be written as
\[
\left( B^{\psi\alpha,q}f \right)(b_m,n,a_m) = \left\langle f, \psi_{b_0;m,n}^{\alpha,q} \right\rangle_{\alpha,q}, \quad m \in \mathbb{Z}, \; n \in \mathbb{N}_0.
\]

The stability condition for this reconstruction takes the form
\[
P \| f \|_{2,\alpha,q}^2 \leq \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \left| \left\langle f, \psi_{b_0;m,n}^{\alpha,q} \right\rangle_{\alpha,q} \right|^2 \leq Q \| f \|_{2,\alpha,q}^2, \; f \in L^2_{\alpha,q}(\mathbb{R}^+),
\]
for certain positive constants \( P \) and \( Q \) satisfying \( 0 < P \leq Q < \infty \).

**Theorem 8.3.** Assume that the discrete \( q \)-Bessel wavelet transform of any \( f \in L^2_{\alpha,q}(\mathbb{R}^+) \) is defined by (8.12) holds. Let \( T \) be a linear operator on \( L^2_{\alpha,q}(\mathbb{R}^+) \) defined by
\[
Tf = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \left\langle f, \psi_{b_0;m,n}^{\alpha,q} \right\rangle_{\alpha,q} \psi_{b_0;m,n}^{\alpha,q},
\]
then
\[
f = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \left\langle f, \psi_{b_0;m,n}^{\alpha,q} \right\rangle_{\alpha,q} \psi_{b_0;m,n}^{\alpha,q},
\]
where
\[
\psi_{\alpha,q,b_0}^{m,n} = T^{-1} \psi_{b_0;m,n}^{\alpha,q}, \; m \in \mathbb{Z}.
\]

**Proof.** From the stability condition (8.12) it follows that defined by (8.13) is a one-one bounded linear operator.

Set
\[
g = Tf, \; f \in L^2_{\alpha,q}(\mathbb{R}^+).
\]

Then we have
\[
\langle Tf, f \rangle_{\alpha,q} = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \left| \left\langle f, \psi_{b_0;m,n}^{\alpha,q} \right\rangle_{\alpha,q} \right|^2.
\]

Therefore,
\[
P \| T^{-1}g \|_{2,\alpha,q}^2 = P \| f \|_{2,\alpha,q}^2 \langle Tf, f \rangle_{\alpha,q}
= \langle g, T^{-1}g \rangle_{\alpha,q}
\leq \| g \|_{2,\alpha,q} \| T^{-1}g \|_{2,\alpha,q},
\]
so that
\[
\| T^{-1}g \|_{\alpha,q} \leq \frac{1}{P} \| g \|_{2,\alpha,q}.
\]

Hence, every \( f \in L^2_{\alpha,q}(\mathbb{R}^+) \) can be reconstructed from its discrete \( q \)-Bessel wavelet transform values given by (8.11).

Thus
\[
f = T^{-1}Tf = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \left\langle f, \psi_{b_0;m,n}^{\alpha,q} \right\rangle_{\alpha,q} T^{-1} \psi_{b_0;m,n}^{\alpha,q}.
\]
Finally, set

\[ \psi_{m,n}^{\alpha,q,b_0} = T^{-1} \psi_{b_0;m,n}^{\alpha,q}, \quad m \in \mathbb{Z}, \quad n \in \mathbb{N}_0. \]  

(8.20)

Then the reconstruction formula (8.19) can be expressed as follows:

\[ f = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \left\langle f, \psi_{b_0;m,n}^{\alpha,q} \right\rangle_{\alpha,q} \psi_{m,n}^{\alpha,q,b_0}. \]

□

References