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Nonparametric Estimation of the Copula Function with Bivariate Twice Censored Data

Toumi Samia, Boukeloua Mohamed*, Idiou Nesrine and Benatia Fatah

abstract: In this work, we are interested in the nonparametric estimation of the copula function in the presence of bivariate twice censored data. Assuming that the copula functions of the right and the left censoring variables are known, we propose an estimator of the joint distribution function of the variables of interest, then we derive an estimator of their copula function. Using a representation of the proposed estimator of the joint distribution function as a sum of independent and identically distributed variables, we establish the weak convergence of the empirical copula that we introduce.

Key Words: Copulas, empirical copula process, twice censored data, product-limit estimator, weak convergence.

Contents

1. Introduction

The study of the dependence between two random variables represents a very important issue in statistics. Among the tools that measure the dependence between two variables, copulas constitute a very useful one because many other tools can be written in terms of copulas, such as the Kendall's tau, the Spearman's rho, the Pearson correlation coefficients and the mutual information. Moreover, copulas characterize the dependence structure between two variables separately from their marginal distributions. The copula C of a couple of real random variables (r.r.v.) $X = (X_1, X_2)$, with a joint distribution function F and continuous margins F_{X_1} and F_{X_2}, is defined on $[0,1]^2$ by $C(u,v) = F\left(F_{X_1}^{-1}(u), F_{X_2}^{-1}(v)\right)$, where $\varphi^{-1}(u) = \inf\{x \in \mathbb{R} : \varphi(x) \geq u\}$ is the generalized inverse of a non decreasing function φ. Sklar's theorem (see [38]) shows that for all $(x_1,x_2) \in \mathbb{R}^2$, $F(x_1,x_2) = C\left(F_{X_1}(x_1), F_{X_2}(x_2)\right)$. The nonparametric estimation of the copula function has been considered for the first time by [10] who introduced the empirical copula. This latter has been widely studied in the literature, we cite for instance the works of [11], [12], [13] and [8]. Moreover, [15] established the weak convergence of this estimator. However, the empirical copula is based on a sample comprising true realizations of the variable of interest, i.e., complete data; but in the practice, one or more censoring phenomena may prevent the observation of the variable of interest and provide only a partial information about it. For example, in the case of right censoring, when a data is censored, the statistician only knows that the variable of interest is greater than the observed value. Bivariate right censored data have been extensively studied in the literature, given their usefullness in the practice; we quote for instance the works of [32], [27], [2] and [20]. The empirical copula for bivariate right censored data has been introduced by [19] for some particular models. Its weak convergence is also established in the same paper. Other copula models for bivariate right censored data have been studied in [36], [34], [6], [23], [22] and [21].

Althoug the right censoring is the most popular type of censored data, more complicated situations can also be encountered in the practice involving right and left censoring at the same time. [33] considered

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one of these situations where the variable of interest is right censored by another variable, the minimum of the two variables is itself left censored and the three latent variables are independent. This is the so called twice censored data model. [31] dealt with a practical situation that corresponds to this model. Moreover, [33] proposed and established the asymptotic properties of a product-limit estimator of the survival function under this model. Then, after the pioneer paper of [33], many authors study the model of twice censored data in the univariate case such as [26], [5] and [7] as well as in the conditional case such as [30] and [3]. In the present paper, we are interested in the nonparametric estimation of the joint distribution function and the copula function of a couple of r.r.v. $X = (X_1, X_2)$, where X_1 and X_2 are both twice censored. For that, we draw on the work of [19]. So, we consider a situation that corresponds to one of the three models studied in this paper, by assuming that the copula functions of the right and the left censoring variables are known. This assumption holds for example when the right (resp. left) censoring variables are independent. Under this assumption, we propose a nonparametric estimator F_n of the joint distribution function of X. Then, we derive from this latter the empirical copula C_n that we propose as an estimator of the copula function. In the case of bivariate right censored data, [19] proved the weak convergence of the empirical copula using a representation of the corresponding estimator of the joint distribution function as a sum of independent and identically distributed (i.i.d.) centered random variables. Such a representation was established by [28]. For our part, we first extend the result of [28] to the case of bivariate twice censored data by representing F_n as a sum of i.i.d. centered random variables. Then, we use this representation to prove the weak convergence of the estimators F_n and C_n (as in [19]). Towards our aim, we introduce the empirical copula for twice censored data in Section 2. In Section 3, we present our assumptions and results and in Section 4, we give some conclusions and perspectives. The proofs of our results are postponed to Appendix A.

2. Empirical copula for twice censored data

Let $X = (X_1, X_2)$ ba a couple of positive r.r.v. with support $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ and joint distribution function F, and let $R = (R_1, R_2)$ (resp. $L = (L_1, L_2)$) be a couple of positive right (resp. left) censoring variables. We assume that the variables X, L and R are independent. In the twice censoring framework, instead of observing X, we observe the independent copies $(Z_{1i}, Z_{2i}, A_{1i}, A_{2i})_{1\le i\le n}$ of the vector (Z_1, Z_2, A_1, A_2), where for $k \in \{1, 2\}$, $Z_k = \max(\min(X_k, R_k), L_k)$ and A_k is the indicator of censoring given by

$$A_k = \begin{cases} 0 & \text{if } \; L_k < X_k \le R_k, \\ 1 & \text{if } \; L_k < R_k < X_k, \\ 2 & \text{if } \; \min(X_k, R_k) \le L_k. \end{cases}$$

In all the sequel, for any r.r.v. V, F_V, S_V, I_V and T_V denote, respectively, the distribution function, the survival function, the lower and the upper endpoint of the support of V. Moreover, for any right continuous function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, we set $\varphi(t^-) = \lim_{\varepsilon \overset{>}{\to} 0} \varphi(t - \varepsilon)$ the left-hand limit of φ at t when it exists. Furthermore, for any differentiable function $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$, we denote by $\partial_1\psi$ (resp. $\partial_2\psi$) the partial derivative of ψ with respect to its first (resp. second) variable.

We assume that the copula function C of X is twice continuously differentiable on $[0, 1]^2$. Furthermore, following [28], we assume that the copula function C_L of L and the survival copula function [1] $\widetilde{C}_R$ of R are known and twice continuously differentiable on $[0, 1]^2$. We also assume that the functions F_{X_k}, F_{R_k} and F_{L_k} $(k \in \{1, 2\})$ are continuous.

To define the empirical copula $C_n(u, v)$ for $(u, v) \in [0, 1]^2$, we need to introduce the following notations. For $k \in \{1, 2\}$ and $j \in \{0, 1, 2\}$, denote by $H_k^{(j)}(t) = P(Z_k \le t, A_k = j)$ the sub-distribution function of Z_k for $A_k = j$, $I_{H_k^{(j)}} = \inf\{t \in \mathbb{R}/H_k^{(j)}(t) > 0\}$ the lower endpoint of the support of $H_k^{(j)}$ and

$$H_{nk}^{(j)}(t) = \frac{1}{n}\sum_{i=1}^{n} I_{\{Z_{ki}\le t, A_{ki}=j\}} \;\; \text{and} \;\; \widehat{F}_{Z_k}(t) = \frac{1}{n}\sum_{i=1}^{n} I_{\{Z_{ki}\le t\}}$$

[1] The survival copula $\widetilde{C}_R$ of R is defined by $\widetilde{C}_R(u, v) = u + v - 1 + C_R(1 - u, 1 - v)$, where C_R is the copula function of R.

($I_{\{.\}}$ being the indicator function) the empirical versions of $H_k^{(j)}$ and F_{Z_k}, respectively.
Furthermore, denote by $\left(Z'_{kj}\right)_{1\leq j\leq m}$ $(m \leq n)$ the distinct values of $(Z_{ki})_{1\leq i\leq n}$. The product-limit estimator $\widehat{F}_{L_k}$ of F_{L_k} is defined by

$$\widehat{F}_{L_k}(t) = \prod_{j/Z'_{kj}>t}\left(1 - \frac{\sum_{i=1}^{n} I_{\left\{Z_{ki}=Z'_{kj}, A_{ki}=2\right\}}}{n\widehat{F}_{Z_k}\left(Z'_{kj}\right)}\right),$$

this estimator can be derived from the Kaplan-Meier one by reversing time.
In addition, the product-limit estimator of S_{R_k} is given by [33] as follows.

$$\widehat{S}_{R_k}(t) = \prod_{i/Z_{ki}\leq t}\left(1 - \frac{I_{\left\{A_{k_i}=1\right\}}}{n\left(\widehat{F}_{L_k}\left(Z_{ki}^{-}\right) - \widehat{F}_{Z_k}\left(Z_{ki}^{-}\right)\right)}\right).$$

Since X is not observed, the empirical distribution function

$$\widetilde{F}_n(x_1, x_2) = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_{1i}\leq x_1, X_{2i}\leq x_2\}}$$

can not be used to estimate $F(x_1, x_2)$. So, following [28] and remarking that

$$E\left[g\left(Z_1, Z_2\right) I_{\{A_1=0\}} I_{\{A_2=0\}} I_{\{Z_1\leq x_1, Z_2\leq x_2\}}\right] = E\left[I_{\{X_1\leq x_1, X_2\leq x_2\}}\right] = F(x_1, x_2),$$

where $g\left(z_1, z_2\right) = C_L\left(F_{L_1}\left(z_1\right), F_{L_2}\left(z_2\right)\right)^{-1} \widetilde{C}_R\left(S_{R_1}\left(z_1\right), S_{R_2}\left(z_2\right)\right)^{-1}$, we propose to replace $I_{\{X_{1i}\leq x_1, X_{2i}\leq x_2\}}$ by the observed quantity

$$\widehat{g}(Z_{1i}, Z_{2i}) I_{\{A_{1i}=0\}} I_{\{A_{2i}=0\}} I_{\{Z_{1i}\leq x_1, Z_{2i}\leq x_2\}},$$

where $\widehat{g}(z_1, z_2) = C_L\left(\widehat{F}_{L_1}\left(z_1\right), \widehat{F}_{L_2}\left(z_2\right)\right)^{-1} \widetilde{C}_R\left(\widehat{S}_{R_1}\left(z_1\right), \widehat{S}_{R_2}\left(z_2\right)\right)^{-1}$.
This gives the following estimator of $F(x_1, x_2)$

$$F_n(x_1, x_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{I_{\{A_{1i}=0\}} I_{\{A_{2i}=0\}}}{C_L\left(\widehat{F}_{L_1}\left(Z_{1i}\right), \widehat{F}_{L_2}\left(Z_{2i}\right)\right) \widetilde{C}_R\left(\widehat{S}_{R_1}\left(Z_{1i}\right), \widehat{S}_{R_2}\left(Z_{2i}\right)\right)} I_{\{Z_{1i}\leq x_1, Z_{2i}\leq x_2\}}.$$

Using this estimator, we propose to estimate $C(u, v)$ as in [19] (relation (3.3)) by

$$C_n\left(u, v\right) = \frac{1}{n}\sum_{i=1}^{n} \frac{I_{\{A_{1i}=0\}} I_{\{A_{2i}=0\}}}{C_L\left(\widehat{F}_{L_1}\left(Z_{1i}\right), \widehat{F}_{L_2}\left(Z_{2i}\right)\right) \widetilde{C}_R\left(\widehat{S}_{R_1}\left(Z_{1i}\right), \widehat{S}_{R_2}\left(Z_{2i}\right)\right)} I_{\{F_{n1}(Z_{1i})\leq u,\, F_{n2}(Z_{2i})\leq v\}},$$

where $F_{n1}(x_1) = \lim_{x_2\to\infty} F_n(x_1, x_2)$ and $F_{n2}(x_2) = \lim_{x_1\to\infty} F_n(x_1, x_2)$.

3. Main results

In this section, we establish the weak convergence of the processes $\sqrt{n}\left(F_n(x_1, x_2) - F(x_1, x_2)\right)$, $(x_1, x_2) \in \mathfrak{X}$ and $\sqrt{n}\left(C_n(u, v) - C(u, v)\right)$, $(u, v) \in [0, 1]^2$. Our approach will be based, as in [19], on a representation of $F_n - F$ as a sum of i.i.d. random variables. So, we will first establish this representation. For that, we need to represent $\widehat{g} - g$ as a sum of i.i.d. random variables. In order to prove such a representation, we begin by introducing some assumptions and notations. For $k \in \{1, 2\}$, denote by $\mathbb{S}_k = \{z \in \mathbb{R} : I_{H_k^{(1)}} < z < \tau_k\}$, where τ_k is such that $I_{H_k^{(1)}} < \tau_k < T_{Z_k}$ and let $\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$. We assume that

H1 $I_{L_k} < I_{R_k}$ and $T_{L_k} < T_{R_k} \leq T_{X_k}$.

H2 There exist $\theta_{k1} > I_{R_k}$ and $\theta_{k2} < T_{R_k}$ such that
$\forall n \in \mathbb{N}^*, \forall 1 \leq i \leq n : A_{ki} = 1 \Rightarrow \theta_{k1} \leq Z_{ki} \leq \theta_{k2}$ almost surely ($a.s.$).

H3 $\int_{I_{H_k^{(1)}}}^{+\infty} \frac{dH_k^{(2)}(z)}{\left(F_{Z_k}(z)\right)^2} < +\infty.$

Assumptions **H1** and **H2** are standard in the twice censoring setting (see e.g. [33], [30] and [24]). Assumption **H3** is needed to obtain the weak convergence of $\sqrt{n}\left(\widehat{F}_{L_k} - F_{L_k}\right)$ and $\sqrt{n}\left(\widehat{S}_{R_k} - S_{R_k}\right)$ on $\mathbb{S}_k$ (see [[33], Lemma 7.2 and Theorem 7.3]). This weak convergence ensures that $\sup_{z\in\mathbb{S}_k}\left|\widehat{F}_{L_k}(z) - F_{L_k}(z)\right| = O_P\left(n^{-1/2}\right)$ and $\sup_{z\in\mathbb{S}_k}\left|\widehat{S}_{R_k}(z) - S_{R_k}(z)\right| = O_P\left(n^{-1/2}\right)$ [2]. So, as in [28], we can use a Taylor expansion to get

$$\widehat{g}(z_1,z_2) - g(z_1,z_2) = -\sum_{k=1,2}\Bigg(C_L\left(F_{L_1}(z_1), F_{L_2}(z_2)\right)\frac{\partial_k\widetilde{C}_R\left(S_{R_1}(z_1), S_{R_2}(z_2)\right)}{\widetilde{C}_R\left(S_{R_1}(z_1), S_{R_2}(z_2)\right)^2}\left(\widehat{S}_{R_k}(z_k) - S_{R_k}(z_k)\right)$$
$$+\widetilde{C}_R\left(S_{R_1}(z_1), S_{R_2}(z_2)\right)\frac{\partial_k C_L\left(F_{L_1}(z_1), F_{L_2}(z_2)\right)}{C_L\left(F_{L_1}(z_1), F_{L_2}(z_2)\right)^2}\left(\widehat{F}_{L_k}(z_k) - F_{L_k}(z_k)\right)\Bigg)$$
$$+ r_n(z_1,z_2), \tag{3.1}$$

where $\sup_{(z_1,z_2)\in\mathbb{S}} |r_n(z_1,z_2)| = o_P\left(n^{-1/2}\right)$.

It remains to represent $\widehat{F}_{L_k} - F_{L_k}$ and $\widehat{S}_{R_k} - S_{R_k}$ as a sum of i.i.d. random variables. The representation of $\widehat{F}_{L_k} - F_{L_k}$ can be deduced from [[29], Theorem 1] by reversing time. In fact, we get for $\delta \in]0,1[$, $I \in \mathbb{R}$ such that $F_{L_k}(I) > \delta$ and $u > I_{Z_k}$
$\widehat{F}_{L_k}(u) - F_{L_k}(u) = F_{L_k}(u)\left(A_{L_k}(n,u) + B_{L_k}(n,u)\right) + R_{L_k}(n,u)$, where

$$A_{L_k}(n,u) = -\frac{H_{nk}^{(2)}(u) - H_k^{(2)}(u)}{F_{Z_k}(u)} - \int_u^{+\infty}\frac{H_{nk}^{(2)}(y) - H_k^{(2)}(y)}{\left(F_{Z_k}(y)\right)^2}dF_{Z_k}(y),$$

$$B_{L_k}(n,u) = \int_u^{+\infty}\frac{\widehat{F}_{Z_k}(y) - F_{Z_k}(y)}{\left(F_{Z_k}(y)\right)^2}dH_k^{(2)}(y)$$

and $R_{L_k}(n,u)$ satisfies

$$\sup_{u\geq I}|R_{L_k}(n,u)| = O_p\left(\frac{1}{n}\right). \tag{3.2}$$

So,

$$\widehat{F}_{L_k}(u) - F_{L_k}(u) = \frac{1}{n}\sum_{i=1}^{n}\Bigg[-\frac{F_{L_k}(u)}{F_{Z_k}(u)}\left(I_{\{Z_{ki}\leq u, A_{ki}=2\}} - H_k^{(2)}(u)\right)$$
$$-F_{L_k}(u)\int_u^{+\infty}\frac{I_{\{Z_{ki}\leq y, A_{ki}=2\}} - H_k^{(2)}(y)}{\left(F_{Z_k}(y)\right)^2}dF_{Z_k}(y)$$
$$+F_{L_k}(u)\int_u^{+\infty}\frac{I_{\{Z_{ki}\leq y\}} - F_{Z_k}(y)}{\left(F_{Z_k}(y)\right)^2}dH_k^{(2)}(y)\Bigg] + R_{L_k}(n,u) \tag{3.3}$$

[2] For a sequence of r.r.v. (ζ_n) and a sequence of non-zero real numbers (u_n), $\zeta_n = O_p(u_n)$ means that $\frac{\zeta_n}{u_n}$ is bounded in probability and $\zeta_n = o_p(u_n)$ means that $\frac{\zeta_n}{u_n}$ converges in probability to zero.

which is a representation of $\widehat{F}_{L_k} - F_{L_k}$ as a sum of i.i.d. centered random variables.
Regarding $\widehat{S}_{R_k} - S_{R_k}$, we give its representation in the following lemma.

Lemma 3.1. *Assume that assumptions* **H1-H3** *hold and let* $\delta \in]0,1[$, $I,\ T \in \mathbb{R}$ *such that* $F_{L_k}(I)\, S_{X_k}(T)\, S_{R_k}(T) > \delta$. *We have*

$$\widehat{S}_{R_k}(u) - S_{R_k}(u) = S_{R_k}(u)\left(A_k(n,u) + B_k(n,u)\right) + R_k(n,u),$$

where

$$A_k(n,u) = -\frac{H_{nk}^{(1)}(u) - H_k^{(1)}(u)}{F_{L_k}(u) - F_{Z_k}(u)} + \int_0^u \frac{H_{nk}^{(1)}(y) - H_k^{(1)}(y)}{\left(F_{L_k}(y) - F_{Z_k}(y)\right)^2} d\left(F_{L_k}(u) - F_{Z_k}(u)\right),$$

$$\begin{aligned} B_k(n,u) = \frac{1}{n}\sum_{i=1}^n \int_0^u &\left\{ \frac{F_{L_k}(y)}{F_{Z_k}(y)}\left(I_{\{Z_{ki}<y, A_{ki}=2\}} - H_k^{(2)}(y)\right) + F_{L_k}(y)\int_y^{+\infty} \frac{I_{\{Z_{ki}\le t, A_{ki}=2\}} - H_k^{(2)}(t)}{\left(F_{Z_k}(t)\right)^2} dF_{Z_k}(t) \right. \\ &\left. -F_{L_k}(y)\int_y^{+\infty} \frac{I_{\{Z_{ki}\le t\}} - F_{Z_k}(t)}{\left(F_{Z_k}(t)\right)^2} dH_k^{(2)}(t) \right\} \frac{dH_k^{(1)}(y)}{\left(F_{L_k}(y) - F_{Z_k}(y)\right)^2} + \int_0^u \frac{\widehat{F}_{Z_k}(y^-) - F_{Z_k}(y)}{\left(F_{L_k}(y) - F_{Z_k}(y)\right)^2} dH_k^{(1)}(y), \end{aligned}$$

and $R_k(n,u)$ *satisfies* $\sup_{I\le u\le T} |R_k(n,u)| = O_P\left(\frac{1}{n}\right)$.

From this lemma, we deduce that

$$\begin{aligned} \widehat{S}_{R_k}(u) - S_{R_k}(u) = \frac{S_{R_k}(u)}{n}\sum_{i=1}^n &\left[\frac{1}{F_{L_k}(u) - F_{Z_k}(u)}\left(H_k^{(1)}(u) - I_{\{Z_{ki}\le u, A_{ki}=1\}}\right) \right. \\ &+ \int_0^u \frac{I_{\{Z_{ki}\le y, A_{ki}=1\}} - H_k^{(1)}(y)}{\left(F_{L_k}(y) - F_{Z_k}(y)\right)^2} d\left(F_{L_k}(y) - F_{Z_k}(y)\right) \\ &+ \int_0^u \left\{ \frac{F_{L_k}(y)}{F_{Z_k}(y)}\left(I_{\{Z_{ki}<y, A_{ki}=2\}} - H_k^{(2)}(y)\right) \right. \\ &+ F_{L_k}(y)\int_y^{+\infty} \frac{I_{\{Z_{ki}\le t, A_{ki}=2\}} - H_k^{(2)}(t)}{\left(F_{Z_k}(t)\right)^2} dF_{Z_k}(t) \\ &\left. - F_{L_k}(y)\int_y^{+\infty} \frac{I_{\{Z_{ki}\le t\}} - F_{Z_k}(t)}{\left(F_{Z_k}(t)\right)^2} dH_k^{(2)}(t) \right\} \frac{dH_k^{(1)}(y)}{\left(F_{L_k}(y) - F_{Z_k}(y)\right)^2} \\ &\left. + \int_0^u \frac{I_{\{Z_{ki}<y\}} - F_{Z_k}(y)}{\left(F_{L_k}(y) - F_{Z_k}(y)\right)^2} dH_k^{(1)}(y) \right] + R_k(n,u) \end{aligned} \tag{3.4}$$

which is a representation of $\widehat{S}_{R_k} - S_{R_k}$ as a sum of i.i.d. centered random variables.
Relations (3.1), (3.3) and (3.4) permit to write

$$\widehat{g}(z_1,z_2) - g(z_1,z_2) = \frac{1}{n}\sum_{i=1}^n \rho(Z_{1i}, Z_{2i}, A_{1i}, A_{2i}; z_1, z_2) + \widetilde{r}_n(z_1,z_2), \tag{3.5}$$

where

$$\rho(Z_{1i},Z_{2i},A_{1i},A_{2i};z_1,z_2)=-\sum_{k=1,2}\left[\left(S_{R_k}(u)\,C_L\left(F_{L_1}(z_1),F_{L_2}(z_2)\right)\frac{\partial_k\widetilde{C}_R\left(S_{R_1}(z_1),S_{R_2}(z_2)\right)}{\widetilde{C}_R\left(S_{R_1}(z_1),S_{R_2}(z_2)\right)^2}\right)\right.$$

$$\times\left(\frac{1}{F_{L_k}(u)-F_{Z_k}(u)}\left(H_k^{(1)}(u)-I_{\{Z_{ki}\le u,A_{ki}=1\}}\right)+\int_0^u\frac{I_{\{Z_{ki}\le y,A_{ki}=1\}}-H_k^{(1)}(y)}{\left(F_{L_k}(y)-F_{Z_k}(y)\right)^2}d\left(F_{L_k}(y)-F_{Z_k}(y)\right)\right.$$

$$+\int_0^u\left\{\frac{F_{L_k}(y)}{F_{Z_k}(y)}\left(I_{\{Z_{ki}<y,A_{ki}=2\}}-H_k^{(2)}(y)\right)+F_{L_k}(y)\int_y^{+\infty}\frac{I_{\{Z_{ki}\le t,A_{ki}=2\}}-H_k^{(2)}(t)}{\left(F_{Z_k}(t)\right)^2}dF_{Z_k}(t)\right.$$

$$\left.\left.-F_{L_k}(y)\int_y^{+\infty}\frac{I_{\{Z_{ki}\le t\}}-F_{Z_k}(t)}{\left(F_{Z_k}(t)\right)^2}dH_k^{(2)}(t)\right\}\frac{dH_k^{(1)}(y)}{\left(F_{L_k}(y)-F_{Z_k}(y)\right)^2}+\int_0^u\frac{I_{\{Z_{ki}<y\}}-F_{Z_k}(y)}{\left(F_{L_k}(y)-F_{Z_k}(y)\right)^2}dH_k^{(1)}(y)\right)$$

$$+\widetilde{C}_R\left(S_{R_1}(z_1),S_{R_2}(z_2)\right)\frac{\partial_k C_L\left(F_{L_1}(z_1),F_{L_2}(z_2)\right)}{C_L\left(F_{L_1}(z_1),F_{L_2}(z_2)\right)^2}\times\left(-\frac{F_{L_k}(u)}{F_{Z_k}(u)}\left(I_{\{Z_{ki}\le u,A_{ki}=2\}}-H_k^{(2)}(u)\right)\right.$$

$$\left.\left.-F_{L_k}(u)\int_u^{+\infty}\frac{I_{\{Z_{ki}\le y,A_{ki}=2\}}-H_k^{(2)}(y)}{\left(F_{Z_k}(y)\right)^2}dF_{Z_k}(y)+F_{L_k}(u)\int_u^{+\infty}\frac{I_{\{Z_{ki}\le y\}}-F_{Z_k}(y)}{\left(F_{Z_k}(y)\right)^2}dH_k^{(2)}(y)\right)\right]$$

and $\sup_{(z_1,z_2)\in\mathbb{S}}|\widetilde{r}_n(z_1,z_2)|=o_P(n^{-1/2})$.
Note that ρ satisfies assumption 2 of [28]. In fact, it is not difficult to check that ρ is centered and that is uniformly bounded on $\mathbb{S}$ under **H1**. Moreover, one can proceed as in Lemma 7.3. of [28] to show that there exists a Donsker class of functions $\mathcal{G}$ such that the function $\frac{1}{n}\sum_{i=1}^n\rho(Z_{1i},Z_{2i},A_{1i},A_{2i};z_1,z_2)$ belongs to $\mathcal{G}$.
Using relation (3.5), we will represent F_n-F as a sum of i.i.d. centered random variables. For that, we need to introduce more notations and assumptions. For any nonempty set A, we denote by $l^\infty(A)$ the space of all bounded real-valued functions defined on A. Moreover, For $k\in\{1,2\}$, denote by

$$\widehat{\Lambda}_{R_k}(t)=\int_0^t\frac{dH_{nk}^{(1)}(u)}{\widehat{F}_{L_k}(u^-)-\widehat{F}_{Z_k}(u^-)}$$

the estimator of the cumulative hazard function Λ_{R_k} of R_k. Thanks to [[33], Theorem 7.3], the process $\sqrt{n}\left(\widehat{\Lambda}_{R_k}(t)-\Lambda_{R_k}(t)\right),t\in\mathbb{S}_k$, converges weakly, under **H1** and **H3** to a centered Gaussian process G_{R_k}. For $s,t\in\mathbb{S}_k$ such that $s\le t$, we denote by

$$\mathcal{K}_{R_k}(s)=cov\left(G_{R_k}(s),G_{R_k}(t)\right)$$

and by

$$\mathcal{K}_{L_k}(s)=\int_s^{+\infty}\frac{dF_{L_k}(u)}{F_{L_k}^2(u)\,F_{X_k\wedge R_k}(u)}$$

the covariance function of the limiting process of the Nelson-Aalan estimator of the cumulative hazard function of L_k (see [[9], Theorem 4] in reversing time). To prove our claimed result, we need the following assumptions which correspond to assumptions 3-5 of [28] adapted to the twice censored data model.

H4 The first and the second partial derivatives of C_L and $\widetilde{C}_R$ are bounded on $[0,1]^2$. Moreover, $C_L(x_1,x_2)\neq0$ and $\widetilde{C}_R(x_1,x_2)\neq0$ for $x_1\neq0$ and $x_2\neq0$.

H5 For $k\in\{1,2\}$, there exist $0\leqslant\alpha_k\leqslant1$ and $0\leqslant\beta_k\leqslant1$ such that

$$C_L(x_1,x_2)\geqslant x_1^{\alpha_1}x_2^{\alpha_2}\quad\text{and}\quad\widetilde{C}_R(x_1,x_2)\geqslant x_1^{\beta_1}x_2^{\beta_2}.$$

H6

$$\int \frac{dF(z_1,z_2)}{C_L(F_{L_1}(z_1),F_{L_2}(z_2))\widetilde{C}_R(S_{R_1}(z_1),S_{R_2}(z_2))}<\infty$$

and for some $\varepsilon>0$ arbitrary small

$$\int\left[\frac{F_{L_1}^{1-\alpha_1}(z_1)\mathcal{K}_{L_1}^{1/2+\varepsilon}(z_1)}{F_{L_2}^{\alpha_2}(z_2)S_{R_1}^{\beta_1}(z_1)S_{R_2}^{\beta_2}(z_2)}+\frac{F_{L_2}^{1-\alpha_2}(z_2)\mathcal{K}_{L_2}^{1/2+\varepsilon}(z_2)}{F_{L_1}^{\alpha_1}(z_1)S_{R_1}^{\beta_1}(z_1)S_{R_2}^{\beta_2}(z_2)}+\frac{S_{R_1}^{1-\beta_1}(z_1)\mathcal{K}_{R_1}^{1/2+\varepsilon}(z_1)}{F_{L_1}^{\alpha_1}(z_1)F_{L_2}^{\alpha_2}(z_2)_2S_{R_2}^{\beta_2}(z_2)}\right.$$
$$\left.+\frac{S_{R_2}^{1-\beta_2}(z_2)\mathcal{K}_{R_2}^{1/2+\varepsilon}(z_2)}{F_{L_1}^{\alpha_1}(z_1)F_{L_2}^{\alpha_2}(z_2)S_{R_1}^{\beta_1}(z_1)}\right]dF(z_1,z_2)<\infty.$$

Theorem 3.2. *Under assumptions* **H1-H6**, *we have*

i) For all $(x_1,x_2)\in\mathcal{X}$

$$F_n(x_1,x_2)-F(x_1,x_2)=\frac{1}{n}\sum_{i=1}^{n}\int_0^{x_1}\int_0^{x_2}[\rho(Z_{1i},Z_{2i},A_{1i},A_{2i};t_1,t_2)C_L(F_{L_1}(t_1),F_{L_2}(t_2))$$
$$\times\widetilde{C}_R(S_{R_1}(t_1),S_{R_2}(t_2))]dF(t_1,t_2)+R_n(x_1,x_2),$$

where $\sup_{(x_1,x_2)\in\mathcal{X}}|R_n(x_1,x_2)|=o_P\left(n^{-1/2}\right)$.

ii) The process $\sqrt{n}(F_n-F)$ *converges weakly in* $l^\infty(\mathcal{X})$ *to a tight centered Gaussian process* G_F.

Note that *ii*) follows directly from *i*) and allows to prove the next theorem which gives the weak convergence of the process $\sqrt{n}(C_n-C)$.

Theorem 3.3. *Under assumptions* **H1-H6**, *the process* $\sqrt{n}(C_n-C)$ *converges weakly in* $l^\infty\left([0,1]^2\right)$ *to the tight Gaussian process*

$$G(u,v)=G^*(u,v)-\partial_1C(u,v)G^*(u,1)-\partial_2C(u,v)G^*(1,v),$$

where $G^*(u,v)=G_F\left(F_{X_1}^{-1}(u),F_{X_2}^{-1}(v)\right)$.

This result is an extension of [[19], Theorem 2] to our case of bivariate twice censored data.

4. Comments and conclusions

In this work, we introduce the empirical copula function in the case of bivariate twice censored data and we establish its weak convergence. Our approach is based on a representation of the corresponding joint distribution function estimator as a sum of i.i.d. centered random variables. The results we obtain extend those given in [28] and [19] in the setting of bivariate right censored data. We prove our results only in the case where the copula functions of the left and the right censoring variables are known. It would be interesting to consider a general bivariate twice censoring model and to look also at other types of censored data, such as doubly or interval censored data. Our obtained results can also be applied to propose and study smoothed copula estimators for bivariate twice censored data.

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A. Proofs

Proof of Lemma 3.1. We follow the same steps of the proof of [[29], Theorem 1]. Let K, C, λ and δ be some positive universal constants. For $k \in \{1,2\}$, we set

$T_k(u) = \log(S_{R_k}(u))$, $\widehat{T}_k(u) = \log\left(\widehat{S}_{R_k}(u)\right) = \sum_{i=1}^{n} I_{\left\{Z_{ki\leq u}, A_{ki}=1\right\}} \log\left(1 - \frac{1}{n\left(\widehat{F}_{L_k}(Z_{ki}^-) - \widehat{F}_{Z_k}(Z_{ki}^-)\right)}\right)$

and $\widetilde{T}_k(u) = -\sum_{i=1}^{n} \frac{I_{\left\{Z_{ki\leq u}, A_{ki}=1\right\}}}{n\left(\widehat{F}_{L_k}(Z_{ki}^-) - \widehat{F}_{Z_k}(Z_{ki}^-)\right)}$.

Proceeding as in [29], we can show that for $I \leq u \leq T$

$$\widehat{S}_{R_k}(u) - S_{R_k}(u) = S_{R_k}(u)\left(A_k(n,u) + \widetilde{B}_k(n,u)\right) + \widetilde{R}_k(n,u), \tag{A.1}$$

where

$$A_k(n,u) = -\frac{H_{nk}^{(1)}(u) - H_k^{(1)}(u)}{F_{L_k}(u) - F_{Z_k}(u)} + \int_0^u \frac{H_{nk}^{(1)}(y) - H_k^{(1)}(y)}{\left(F_{L_k}(y) - F_{Z_k}(y)\right)^2} d\left(F_{L_k}(u) - F_{Z_k}(u)\right),$$

$$\widetilde{B}_k(n,u) = -\int_0^u \frac{\widehat{F}_{L_k}(y^-) - \widehat{F}_{Z_k}(y^-) - F_{L_k}(y) + F_{Z_k}(y)}{\left(F_{L_k}(y) - F_{Z_k}(y)\right)^2} dH_k^{(1)}(y)$$

and

$$\widetilde{R}_k(n,u) = S_{R_k}(u)\left(R_{k2}(u) + R_{k3}(u) + R_{k4}(u)\right) + R_{k1}(u),$$

with

$$R_{k1}(u) = \widehat{S}_{R_k}(u) - S_{R_k}(u) - S_{R_k}(u)\left(\widehat{T}_k(u) - T_k(u)\right),$$

$$R_{k2}(u) = \widehat{T}_k(u) - \widetilde{T}_k(u),$$

$$R_{k3}(u) = \widetilde{T}_k(u) + \sum_{i=1}^{n} \frac{I_{\left\{Z_{ki\leq u}, A_{ki}=1\right\}}}{n\left(\widehat{F}_{L_k}(Z_{ki}^-) - \widehat{F}_{Z_k}(Z_{ki}^-)\right)} \times \left[1 - \frac{n\left(\widehat{F}_{L_k}(Z_{ki}^-) - \widehat{F}_{Z_k}(Z_{ki}^-) - F_{L_k}(Z_{ki}) + F_{Z_k}(Z_{ki})\right)}{n\left(F_{L_k}(Z_{ki}) - F_{Z_k}(Z_{ki})\right)}\right]$$

and

$$R_{k4}(u)=\frac{1}{n}\int_0^{+\infty}\int_0^{+\infty}\frac{I_{\{y\le u\}}I_{\{x<y\}}}{\left(F_{L_k}(y)-F_{Z_k}(y)\right)^2}d\left[\sqrt{n}\left(H_{nk}^{(1)}(y)-H_k^{(1)}(y)\right)\right]$$
$$d\left[\sqrt{n}\left(\widehat{F}_{L_k}(x^-)-\widehat{F}_{Z_k}(x^-)-F_{L_k}(x)+F_{Z_k}(x)\right)\right].$$

As in [29], we will prove the following lemmas.

Lemma A.1. $\sup_{I\le u\le T}|R_{k2}(u)|=O_{a.s.}\left(\frac{1}{n}\right)$.

Lemma A.2. $P\left(\sup_{I\le u\le T}|nR_{k3}(u)|>x\right)\le K\exp\left\{-\lambda\delta^2 x\right\}$ *if* $0\le x<\frac{2n}{\delta}$.

Lemma A.3. $P\left(\sup_{I\le u\le T}|nR_{k4}(u)|>x\right)\le K\exp\left\{-\lambda\delta^2 x\right\}$ *for* $x>0$.

From these lemmas, we deduce that

$$\sup_{I\le u\le T}\left|\widetilde{R}_k(n,u)\right|=O_P\left(\frac{1}{n}\right). \tag{A.2}$$

Moreover, relation (3.3) permits to write

$$\widetilde{B}_k(n,u)=-\int_0^u\frac{\left(\widehat{F}_{L_k}(y^-)-F_{L_k}(y)\right)}{\left(F_{L_k}(y)-F_{Z_k}(y)\right)^2}dH_k^{(1)}(y)+\int_0^u\frac{\widehat{F}_{Z_k}(y^-)-F_{Z_k}(y)}{\left(F_{L_k}(y)-F_{Z_k}(y)\right)^2}dH_k^{(1)}(y)$$
$$=B_k(n,u)-\int_0^u\frac{R_{L_k}(n,y^-)}{\left(F_{L_k}(y)-F_{Z_k}(y)\right)^2}dH_k^{(1)}(y).$$

Combining this with (A.1), we obtain

$$\widehat{S}_{R_k}(u)-S_{R_k}(u)=S_{R_k}(u)\left(A_k(n,u)+B_k(n,u)\right)+R_k(n,u),$$

where

$$R_k(n,u)=-S_{R_k}(u)\int_0^u\frac{R_{L_k}(n,y^-)}{\left(F_{L_k}(y)-F_{Z_k}(y)\right)^2}dH_k^{(1)}(y)+\widetilde{R}_k(n,u)$$
$$=:\widetilde{\widetilde{R}}_k(n,u)+\widetilde{R}_k(n,u). \tag{A.3}$$

Since $F_{L_k}(y)-F_{Z_k}(y)=F_{L_k}(y)S_{X_k}(y)S_{R_k}(y)$, we get

$$\sup_{I\le u\le T}\left|\widetilde{\widetilde{R}}_k(n,u)\right|\le\sup_{I\le u\le T}|R_{L_k}(n,u)|\int_0^u\frac{dH_k^{(1)}(y)}{\left(F_{L_k}(y)S_{X_k}(y)S_{R_k}(y)\right)^2}$$
$$\le\frac{1}{\left(F_{L_k}(I_{R_k})S_{X_k}(T)S_{R_k}(T)\right)^2}\sup_{I_{R_k}\le u\le T}|R_{L_k}(n,u)|.$$

This together with (3.2), (A.2) and (A.3) permits to write

$$\sup_{I\le u\le T}|R_k(n,u)|=O_P\left(\frac{1}{n}\right),$$

which gives the claimed result. □

It remains to prove lemmas $2-4$.

Proof of Lemma A.1. We have

$$\begin{aligned}
|R_{k_2}(u)| &= \left|\widehat{T}_k(u) - \widetilde{T}_k(u)\right| \\
&\leq \sum_{i=1}^{n}\left| I_{\{Z_{k_i}\leq u, A_{ki}=1\}}\log\left(1-\frac{1}{n\left(\widehat{F}_{L_k}(Z_{k_i}^-)-\widehat{F}_{Z_k}(Z_{k_i}^-)\right)}\right)+\frac{I_{\{Z_{k_i}\leq u, A_{ki}=1\}}}{n\left(\widehat{F}_{L_k}(Z_{k_i}^-)-\widehat{F}_{Z_k}(Z_{k_i}^-)\right)}\right| \\
&\leq \sum_{i=1}^{n}\left|\log\left(1-\frac{1}{n\left(\widehat{F}_{L_k}(Z_{k_i}^-)-\widehat{F}_{Z_k}(Z_{k_i}^-)\right)}\right)+\frac{1}{n\left(\widehat{F}_{L_k}(Z_{k_i}^-)-\widehat{F}_{Z_k}(Z_{k_i}^-)\right)}\right| I_{\{Z_{k_i}\leq T\}}.
\end{aligned}$$

Since $|\log(1-z)+z|\leq z^2$ for $0\leq z\leq 1/2$ and $\inf_{I_{R_k}\leq t\leq T} n\left\{\widehat{F}_{L_k}(t^-)-\widehat{F}_{Z_k}(t^-)\right\} \geq \frac{n}{2}\left(F_{L_k}(I_{R_k})S_{X_k}(T)S_{R_k}(T)\right) \geq 2$ for n large enough, we deduce that for $I\leq u\leq T$

$$\begin{aligned}
|R_{k_2}(u)| &\leq \frac{1}{n^2}\sum_{i=1}^{n}\frac{I_{\{Z_{ki}\leq T\}}}{\left(\widehat{F}_{L_k}(Z_{k_i}^-)-\widehat{F}_{Z_k}(Z_{k_i}^-)\right)^2} \\
&\leq \frac{1}{n}\frac{1}{\inf_{I_{R_k}\leq t\leq T}\left(\widehat{F}_{L_k}(t^-)-\widehat{F}_{Z_k}(t^-)\right)^2} \\
&\leq \frac{1}{n}\frac{4}{\left(F_{L_k}(I_{R_k})S_{X_k}(T)S_{R_k}(T)\right)^2} \quad a.s. \text{ for n large enough.}
\end{aligned}$$

Thus $\sup_{I\leq u\leq T}|R_{k_2}(u)| = O_{a.s.}\left(\frac{1}{n}\right)$. ◻

Proof of Lemma A.2. To prove this lemma, we need to apply some exponential inequalities for $\widehat{F}_{Z_k}$, $H_{nk}^{(1)}$ and $\widehat{F}_{L_k}$. Regarding $\widehat{F}_{Z_k}$, [14] proved that there exists a positive constant D such that for all $x>0$

$$P\left(\sqrt{n}\sup_{t\in\mathbb{R}}\left|\widehat{F}_{Z_k}(t)-F_{Z_k}(t)\right|>x\right)\leq D\exp\left(-2x^2\right). \tag{A.4}$$

Moreover, writing

$$H_{nk}^{(1)}(t)=\frac{1}{n}\sum_{i=1}^{n}I_{\{R_{ki}\leq t,\, L_{ki}-R_{ki}<0,\, R_{ki}-X_{ki}<0\}}$$

allows to apply [[25], Theorem 1-m] to get for all $x>0$ and $\varepsilon>0$

$$P\left(\sqrt{n}\sup_{t\in\mathbb{R}}\left|H_{nk}^{(1)}(t)-H_{k}^{(1)}(t)\right|>x\right)\leq D\exp\left(-(2-\varepsilon)x^2\right), \tag{A.5}$$

where D is a positive constant.
Furthermore, adapting [[4], Theorem 1], we get for all $x>0$ and $\theta>\min(I_{X_k}, I_{R_k})$

$$P\left(\sqrt{n}\sup_{t\geq\theta}\left|\widehat{F}_{L_k}(t)-F_{L_k}(t)\right|>\frac{x}{F_{X_k\wedge R_x}(\theta)}\right)\leq 2.5\exp\left(-2x^2+Dx\right), \tag{A.6}$$

where D is a positive constant.
Set

$$\Gamma_{kn}=\left\{\inf_{\theta_{k1}\leq t\leq\theta_{k2}}\left\{\widehat{F}_{L_k}(t^-)-\widehat{F}_{Z_k}(t^-)\right\}\geq\frac{2}{n}\right\}$$

and

$$\Delta_{kn}(u) = \Big\{ \Big| \widehat{F}_{L_k}(Z_{ki}^-) - \widehat{F}_{Z_k}(Z_{ki}^-) - F_{L_k}(Z_{ki}) + F_{Z_k}(Z_{ki}) \Big| < \frac{1}{2}\left(F_{L_k}(Z_{ki}) - F_{Z_k}(Z_{ki})\right) \text{ or } \\ Z_{ki} \le u \text{ for all } 1 \le i \le n \text{ such that } A_{ki} = 1 \Big\}.$$

Remarking that

$$\widetilde{T}_k(u) = -\sum_{i=1}^{n} \frac{I_{\{Z_{ki} \le u, A_{ki}=1\}}}{n\left(F_{L_k}(Z_{ki}^-) - F_{Z_k}(Z_{ki}^-)\right)} \frac{1}{1 + \frac{n\left(\widehat{F}_{L_k}(Z_{ki}^-) - \widehat{F}_{Z_k}(Z_{ki}^-) - F_{L_k}(Z_{ki}) + F_{Z_k}(Z_{ki})\right)}{n\left(F_{L_k}(Z_{ki}) - F_{Z_k}(Z_{ki})\right)}}$$

and that $\left|\frac{1}{1+z} - 1 + z\right| < 2z^2$ for $|z| < 1/2$, we deduce that on Γ_{kn}, we have

$$|R_{k3}(u)| \le 2 \sum_{i=1}^{n} \frac{I_{\{Z_{ki} \le u, A_{ki}=1\}}}{n\left(F_{L_k}(Z_{ki}) - F_{Z_k}(Z_{ki})\right)} \left[\frac{n\left(\widehat{F}_{L_k}(Z_{ki}^-) - \widehat{F}_{Z_k}(Z_{ki}^-) - F_{L_k}(Z_{ki}) + F_{Z_k}(Z_{ki})\right)}{n\left(F_{L_k}(Z_{ki}) - F_{Z_k}(Z_{ki})\right)} \right]^2.$$

Therefore

$$\begin{aligned} P\left(\sup_{I \le u \le T} |nR_{k3}(u)| > x\right) \le{}& P\left(\Gamma_{kn}^c\right) + P\left(\left(\Delta_{kn}(I)\right)^c\right) \\ &+ P\Bigg[\sup_{I \le u \le T} \Bigg\{ 2 \sum_{i=1}^{n} \frac{I_{\{Z_{ki} \le u, A_{ki}=1\}}}{F_{L_k}(Z_{ki}) - F_{Z_k}(Z_{ki})} \\ &\times \left(\frac{n\left(\widehat{F}_{L_k}(Z_{ki}^-) - \widehat{F}_{Z_k}(Z_{ki}^-) - F_{L_k}(Z_{ki}) + F_{Z_k}(Z_{ki})\right)}{n\left(F_{L_k}(Z_{ki}) - F_{Z_k}(Z_{ki})\right)} \right)^2 \Bigg\} > x \Bigg]. \end{aligned} \tag{A.7}$$

Moreover, we have for n large enough

$$\begin{aligned} &\inf_{\theta_{k1} \le t \le \theta_{k2}} \left\{ \widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-) \right\} < \tfrac{2}{n} \Rightarrow \exists t_0 \in [\theta_{k1}, \theta_{k2}] \text{ such that } \widehat{F}_{L_k}(t_0^-) - \widehat{F}_{Z_k}(t_0^-) < \frac{2}{n} \\ &\Rightarrow \sup_{\theta_{k1} \le t \le \theta_{k2}} \left| \widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-) - F_{L_k}(t) + F_{Z_k}(t) \right| \ge F_{L_k}(t_0) - F_{Z_k}(t_0) - \left(\widehat{F}_{L_k}(t_0^-) - \widehat{F}_{Z_k}(t_0^-)\right) \\ &\qquad \ge F_{L_k}(\theta_{k1}) S_{X_k}(\theta_{k2}) S_{R_k}(\theta_{k2}) - \frac{2}{n} \\ &\qquad > \frac{F_{L_k}(\theta_{k1}) S_{X_k}(\theta_{k2}) S_{R_k}(\theta_{k2})}{2} =: \frac{a}{2}. \end{aligned}$$

So

$$\begin{aligned} P\left(\inf_{\theta_{k1} \le t \le \theta_{k2}} \left(\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)\right) < \frac{2}{n}\right) &\le P\Bigg(\sqrt{n} \sup_{\theta_{k1} \le t \le \theta_{k2}} \Big| \widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-) \\ &\qquad - F_{L_k}(t) + F_{Z_k}(t) \Big| > \frac{a\sqrt{n}}{2} \Bigg) \\ &\le K \exp\{-Cn\} \quad \text{(thanks to relations (A.4) and (A.6))} \\ &\le K \exp\left\{-\lambda \delta^2 x\right\} \text{ for } n \text{ large enough.} \end{aligned} \tag{A.8}$$

Furthermore, we have

$$P\left((\Delta_{kn}(I))^c\right) = P\left(\bigcup_{i=1}^{n}\left(\left\{\frac{\left|\widehat{F}_{L_k}\left(Z_{ki}^-\right) - \widehat{F}_{Z_k}\left(Z_{ki}^-\right) - F_{L_k}\left(Z_{ki}\right) + F_{Z_k}\left(Z_{ki}\right)\right|}{F_{L_k}\left(Z_{ki}\right) - F_{Z_k}\left(Z_{ki}\right)} > 2\right\}\right.\right.$$

$$\left.\left.\cap\left\{Z_{ki} > I, A_{ki} = 1\right\}\right)\right)$$

$$\leq nP\left(\frac{\left|\widehat{F}_{L_k}\left(Z_k^-\right) - \widehat{F}_{Z_k}\left(Z_k^-\right) - F_{L_k}\left(Z_k\right) + F_{Z_k}\left(Z_k\right)\right|}{F_{L_k}\left(Z_k\right) - F_{Z_k}\left(Z_k\right)} > 2, Z_k > I, A_k = 1\right)$$

and for $t > I$, we have

$$P\left(\frac{\left|\widehat{F}_{L_k}\left(Z_k^-\right) - \widehat{F}_{Z_k}\left(Z_k^-\right) - F_{L_k}\left(Z_k\right) + F_{Z_k}\left(Z_k\right)\right|}{F_{L_k}\left(Z_k\right) - F_{Z_k}\left(Z_k\right)} > 2, Z_k > I, A_k = 1\middle| Z_k = t\right)$$

$$= P\left(\frac{\left|\widehat{F}_{L_k}\left(t^-\right) - \widehat{F}_{Z_k}\left(t^-\right) - F_{L_k}\left(t\right) + F_{Z_k}\left(t\right)\right|}{F_{L_k}\left(t\right) - F_{Z_k}\left(t\right)} > 2, Z_k > I, A_k = 1\middle| Z_k = t\right)$$

$$\leq P\left(\left|\widehat{F}_{L_k}\left(t^-\right) - F_{L_k}\left(t\right)\right| > F_{L_k}\left(t\right) - F_{Z_k}\left(t\right), Z_k > I, A_k = 1 \mid Z_k = t\right)$$

$$+ P\left(\left|\widehat{F}_{Z_k}\left(t^-\right) - F_{Z_k}\left(t\right)\right| > F_{L_k}\left(t\right) - F_{Z_k}\left(t\right), Z_k > I, A_k = 1 \mid Z_k = t\right).$$

On the one hand, the Bernstein inequality (see [[16], Corollary $A.9$]) allows to write

$$P\left(\left|\widehat{F}_{Z_k}\left(t^-\right) - F_{Z_k}\left(t\right)\right| > F_{L_k}\left(t\right) - F_{Z_k}\left(t\right), Z_k > I, A_k = 1 \mid Z_k = t\right)$$

$$= P\left(\left|\sum_{j=1}^{n}\left(I_{\{Z_{kj}<t\}} - F_{Z_k}\left(t\right)\right)\right| > n\left(F_{L_k}\left(t\right) - F_{Z_k}\left(t\right)\right), Z_k > I, A_k = 1 \mid Z_k = t\right)$$

$$\leq 2\exp\left\{\frac{-2\left(F_{Z_k}\left(t\right)\right)^2 n}{F_{Z_k}\left(t\right) S_{Z_k}\left(t\right)\left(1 + \frac{4F_{Z_k}(t)}{F_{Z_k}(t)S_{Z_k}(t)}\right)}\right\}$$

$$\leq 2\exp\left\{-CnF_{Z_k}\left(t\right)\right\}$$

and the probability equals to zero if $t \leq I$.
On the other hand, proceeding as in [[35], proof of Theorem 2], we get

$$\sup_{u\geq I}\left|\widehat{F}_{L_k}\left(u^-\right) - F_{L_k}\left(u\right)\right| \leq C\left[\sup_{u\geq I}\left|\widehat{F}_{Z_k}\left(u^-\right) - F_{Z_k}\left(u\right)\right| + \sup_{u\geq I}\left|H_{kn}^{(2)}\left(u^-\right) - H_k^{(2)}\left(u\right)\right|\right].$$

So, for $t > I$

$$P\left(\left|\widehat{F}_{L_k}\left(t^-\right) - \widehat{F}_{Z_k}\left(t^-\right)\right| > F_{L_k}\left(t\right) - F_{Z_k}\left(t\right), Z_k > I, A_k = 1\middle| Z_k = t\right)$$

$$\leq P\left(\sup_{u\geq I}\left|\frac{\widehat{F}_{Z_k}\left(u^-\right) - F_{Z_k}\left(u\right)}{F_{Z_k}\left(u\right)}\right| > \frac{b}{2C}, Z_k > I, A_k = 1\middle| Z_k = t\right)$$

$$+ P\left(\sup_{u\geq I}\left|\frac{H_{kn}^{(2)}\left(u^-\right) - H_k^{(2)}\left(u\right)}{H_k^{(2)}\left(u\right)}\right| > \frac{b}{2C}, Z_k > I, A_k = 1\middle| Z_k = t\right) \text{ (where } b = F_{L_k}(I)S_{X_k}(\theta_{k2})S_{R_k}(\theta_{k2})\text{)}$$

$$\leq K_1\exp\left\{-C_1 n\right\} + K_2\exp\left\{-C_2 n\right\}$$

(thanks to lemma 3 of [39]; K_1, K_2, C_1 and C_2 being some positive constants)

$$\leq K\exp\left\{-CnF_{Z_k}\left(t\right)\right\}$$

and the probability equals to zero if $t \geq I$. Thus

$$P\left((\Delta_{kn}(I))^c\right) \leq Kn \int_I^{+\infty} \exp\left\{-CnF_{Z_k}(t)\right\} dF_{Z_k}(t) \leq K\exp\left\{-\frac{C\delta n}{2}\right\}. \quad (A.9)$$

It remains to deal with the following probability

$$P\left[\sup_{I\leq u\leq T}\left(2\sum_{i=1}^{n}\frac{I_{\{Z_{ki}\leq u, A_{ki}=1\}}}{F_{L_k}(Z_{ki})-F_{Z_k}(Z_{ki})}\left[\frac{n\left(\widehat{F}_{L_k}(Z_{ki}^-)-\widehat{F}_{Z_k}(Z_{ki}^-)-F_{L_k}(Z_{ki})+F_{Z_k}(Z_{ki})\right)}{n\left(F_{L_k}(Z_{ki})-F_{Z_k}(Z_{ki})\right)}\right]^2 > x\right)\right]$$

$$\leq P\left(2\sum_{i=1}^{n}\frac{I_{\{Z_{ki}\leq T, A_{ki}=1\}}}{n\left(F_{L_k}(Z_{ki})-F_{Z_k}(Z_{ki})\right)}\left[\frac{n\left(\widehat{F}_{L_k}(Z_{ki}^-)-\widehat{F}_{Z_k}(Z_{ki}^-)-F_{L_k}(Z_{ki})+F_{Z_k}(Z_{ki})\right)}{\sqrt{n}\left(F_{L_k}(Z_{ki})-F_{Z_k}(Z_{ki})\right)}\right]^2 > x\right)$$

$$\leq P\left(\frac{2}{a^3}\sup_{\theta_{k1}\leq u\leq\theta_{k2}} n\left(\left(\widehat{F}_{L_k}(u^-)-F_{L_k}(u)\right)-\left(\widehat{F}_{Z_k}(u^-)-F_{Z_k}(u)\right)\right)^2 > x\right)$$

$$\leq P\left(\frac{2}{C\delta}\sup_{\theta_{k1}\leq u\leq\theta_{k2}}\left(\sqrt{n}\frac{\widehat{F}_{L_k}(u^-)-F_{L_k}(u)}{F_{L_k}(u)}\right)^2 > \frac{x}{2}\right)$$

$$+ P\left(\frac{2}{C\delta}\sup_{\theta_{k1}\leq u\leq\theta_{k2}}\left(\sqrt{n}\frac{\widehat{F}_{Z_k}(u^-)-F_{Z_k}(u)}{F_{Z_k}(u)}\right)^2 > \frac{x}{2}\right)$$

$$\leq P\left(\frac{2C_1}{C\delta}\sup_{\theta_{k1}\leq u\leq\theta_{k2}}\left(\sqrt{n}\frac{\widehat{F}_{Z_k}(u^-)-F_{Z_k}(u)}{F_{Z_k}(u)}\right)^2 > \frac{x}{4}\right)$$

$$+ P\left(\frac{2C_2}{C\delta}\sup_{\theta_{k1}\leq u\leq\theta_{k2}}\left(\sqrt{n}\frac{H_{kn}^{(2)}(u^-)-H_k^{(2)}(u)}{H_k^{(2)}(u)}\right)^2 > \frac{x}{4}\right)$$

$$+ P\left(\frac{2}{C\delta}\sup_{\theta_{k1}\leq u\leq\theta_{k2}}\left(\sqrt{n}\frac{\widehat{F}_{Z_k}(u^-)-F_{Z_k}(u)}{F_{Z_k}(u)}\right)^2 > \frac{x}{2}\right) \text{ (C_1 and C_2 are positive constants)}$$

$$\leq K\exp\left\{-\lambda\delta^2 x\right\} \text{ (thanks to [[39], Lemma 3]).}$$

Combining this with (A.7),(A.8) and (A.9) gives the claimed result. □

Proof of Lemma A.3. We have $nR_{k4}(u) = J_{k1}(u) + J_{k2}(u)$, where

$$J_{k1}(u) = \int_0^{+\infty}\int_0^{+\infty}\frac{I_{\{y\leq u\}}I_{\{x<y\}}}{\left(F_{L_k}(y)-F_{Z_k}(y)\right)^2} d\left[\sqrt{n}\left(H_{nk}^{(1)}(y)-H_k^{(1)}(y)\right)\right] d\left[\sqrt{n}\left(\widehat{F}_{Z_k}(x^-)-F_{Z_k}(x)\right)\right]$$

and

$$J_{k2}(u) = \int_0^{+\infty}\int_0^{+\infty}\frac{I_{\{y\leq u\}}I_{\{x<y\}}}{\left(F_{L_k}(y)-F_{Z_k}(y)\right)^2} d\left[\sqrt{n}\left(H_{nk}^{(1)}(y)-H_k^{(1)}(y)\right)\right] d\left[\sqrt{n}\left(\widehat{F}_{L_k}(x^-)-F_{L_k}(x)\right)\right].$$

So

$$P\left(\sup_{I\leq u\leq T}|nR_{k4}(u)| > x\right) \leq P\left(\sup_{I\leq u\leq T}|J_{k1}(u)| > \frac{x}{2}\right) + P\left(\sup_{I\leq u\leq T}|J_{k2}(u)| > \frac{x}{2}\right). \quad (A.10)$$

On the one hand, we can prove as in [[29], Lemma 3] that

$$P\left(\sup_{I\leq u\leq T} n|J_{k1}(u)| > \frac{x}{2}\right) \leq K\exp\left\{-\lambda\delta^2 x\right\}. \quad (A.11)$$

On the other hand, we have for $I \leq u \leq T$

$$J_{k2}(u) = \int_0^{+\infty} \frac{\widehat{F}_{L_k}(y^-) - F_{L_k}(y)}{(F_{L_k}(y) - F_{Z_k}(y))^2} I_{\{y \leq u\}} d\left(H_{nk}^{(1)}(y) - H_k^{(1)}(y)\right).$$

Therefore

$$\begin{aligned} |J_{k2}(u)| &\leq \frac{1}{a^2} \sup_{\theta_{k1} \leq u \leq \theta_{k2}} \left|\widehat{F}_{L_k}(u^-) - F_{L_k}(u)\right| \left|H_{nk}^{(1)}(u) - H_k^{(1)}(u)\right| \\ &\leq \frac{1}{a^2} \sup_{\theta_{k1} \leq u \leq \theta_{k2}} \left|\widehat{F}_{L_k}(u^-) - F_{L_k}(u)\right| \sup_{I \leq u \leq T} \left|H_{nk}^{(1)}(u) - H_k^{(1)}(u)\right| \end{aligned}$$

which implies that

$$\begin{aligned} P\left(\sup_{I \leq u \leq T} n |J_{k2}(u)| > \frac{x}{2}\right) &\leq P\left(\sqrt{n} \sup_{\theta_{k1} \leq u \leq \theta_{k2}} \left|\widehat{F}_{L_k}(u^-) - F_{L_k}(u)\right| > \sqrt{\frac{ax}{2}}\right) \\ &\quad + P\left(\sqrt{n} \sup_{I \leq u \leq T} \left|H_{nk}^{(1)}(u) - H_k^{(1)}(u)\right| > \sqrt{\frac{ax}{2}}\right) \\ &\leq K \exp\{-Cx\} \ \ \text{(thanks to relations (A.5) and (A.6))} \\ &\leq K \exp\{-\lambda \delta^2 x\}. \end{aligned}$$

This together with (A.10) and (A.11) gives the claimed result. □

Proof of Theorem 3.2-i). Using the following lemma, theorem 3.2-*i*) can be proved in the same way as in [[28], Theorem 3], for the class of functions

$$\mathcal{F} = \left\{(t_1, t_2) \mapsto I_{[0,x_1]\times[0,x_2]}(t_1, t_2),\ x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2\right\}.$$

□

Lemma A.4. *Under assumptions* **H1-H6**, *we have for all* $\varepsilon > 0$

$$\begin{aligned} &\max_{1 \leq i \leq n} \left| \frac{I_{\{A_{1i}=1\}} I_{\{A_{2i}=1\}}}{C_L\left(\widehat{F}_{L_1}(Z_{1i}), \widehat{F}_{L_2}(Z_{2i})\right) \widetilde{C}_R\left(\widehat{S}_{R_1}(Z_{1i}), \widehat{S}_{R_2}(Z_{2i})\right)} - \frac{I_{\{A_{1i}=1\}} I_{\{A_{2i}=1\}}}{C_L\left(F_{L_1}(Z_{1i}), F_{L_2}(Z_{2i})\right) \widetilde{C}_R\left(S_{R_1}(Z_{1i}), S_{R_2}(Z_{2i})\right)} \right| \\ &\leq \mathcal{M}_n \left(\frac{I_{\{A_{1i}=1\}} I_{\{A_{2i}=1\}} F_{L_1}^{1-\alpha_1}(Z_{1i}) \mathcal{K}_{L_1}^{1/2+\varepsilon}(Z_{1i})}{F_{L_2}^{\alpha_2}(Z_{2i}) S_{R_1}^{\beta_1}(Z_{1i}) S_{R_2}^{\beta_2}(Z_{2i}) C_L(F_{L_1}(Z_{1i}), F_{L_2}(Z_{2i})) \widetilde{C}_R(S_{R_1}(Z_{1i}), S_{R_2}(Z_{2i}))} \right. \\ &+ \frac{I_{\{A_{1i}=1\}} I_{\{A_{2i}=1\}} F_{L_2}^{1-\alpha_2}(Z_{2i}) \mathcal{K}_{L_2}^{1/2+\varepsilon}(Z_{2i})}{F_{L_1}^{\alpha_1}(Z_{1i}) S_{R_1}^{\beta_1}(Z_{1i}) S_{R_2}^{\beta_2}(Z_{2i}) C_L(F_{L_1}(Z_{1i}), F_{L_2}(Z_{2i})) \widetilde{C}_R(S_{R_1}(Z_{1i}), S_{R_2}(Z_{2i}))} \\ &+ \frac{I_{\{A_{1i}=1\}} I_{\{A_{2i}=1\}} S_{R_1}^{1-\beta_1}(Z_{1i}) \mathcal{K}_{R_1}^{1/2+\varepsilon}(Z_{1i})}{F_{L_1}^{\alpha_1}(Z_{1i}) F_{L_2}^{\alpha_2}(Z_{2i}) S_{R_2}^{\beta_2}(Z_{2i}) C_L(F_{L_1}(Z_{1i}), F_{L_2}(Z_{2i})) \widetilde{C}_R(S_{R_1}(Z_{1i}), S_{R_2}(Z_{2i}))} \\ &\left. + \frac{I_{\{A_{1i}=1\}} I_{\{A_{2i}=1\}} S_{R_2}^{1-\beta_2}(Z_{2i}) \mathcal{K}_{R_2}^{1/2+\varepsilon}(Z_{2i})}{F_{L_1}^{\alpha_1}(Z_{1i}) F_{L_2}^{\alpha_2}(Z_{2i}) S_{R_1}^{\beta_1}(Z_{1i}) C_L(F_{L_1}(Z_{1i}), F_{L_2}(Z_{2i})) \widetilde{C}_R(S_{R_1}(Z_{1i}), S_{R_2}(Z_{2i}))} \right), \end{aligned}$$

with $\mathcal{M}_n = O_P\left(n^{-1/2}\right)$.

This lemma is the equivalent of [[28], Lemma 7.2] in the case of twice censoring.

Proof of Lemma A.4. Let $Z_{(kn)}=\max_{k\leq i\leq n} Z_{ki}$. Proceeding as in [[28], Lemma 7.2], we obtain

$$\left|\frac{I_{\{A_{1i}=1\}}\, I_{\{A_{2i}=1\}}}{C_L\left(\widehat{F}_{L_1}(Z_{1i}),\widehat{F}_{L_2}(Z_{2i})\right)\widetilde{C}_R\left(\widehat{S}_{R_1}(Z_{1i}),\widehat{S}_{R_2}(Z_{2i})\right)}-\frac{I_{\{A_{1i}=1\}}\, I_{\{A_{2i}=1\}}}{C_L\left(F_{L_1}(Z_{1i}),F_{L_2}(Z_{2i})\right)\widetilde{C}_R\left(S_{R_1}(Z_{1i}),S_{R_2}(Z_{2i})\right)}\right|$$

$$\leq \frac{M\ I_{\{A_{1i}=1\}}\ I_{\{A_{2i}=1\}}}{C_L\left(F_{L_1}\left(Z_{1i}\right),F_{L_2}\left(Z_{2i}\right)\right)\widetilde{C}_R\left(S_{R_1}\left(Z_{1i}\right),S_{R_2}\left(Z_{2i}\right)\right)}\times\left[\frac{\left|\widehat{F}_{L_1}\left(Z_{1i}\right)-F_{L_1}\left(Z_{1i}\right)\right|}{\widehat{F}_{L_1}^{\alpha_1}\left(Z_{1i}\right)\widehat{F}_{L_2}^{\alpha_2}\left(Z_{2i}\right)\widehat{S}_{R_1}^{\beta_1}\left(Z_{1i}\right)\widehat{S}_{R_2}^{\beta_2}\left(Z_{2i}\right)}\right.$$

$$+\frac{\left|\widehat{F}_{L_2}\left(Z_{2i}\right)-F_{L_2}\left(Z_{2i}\right)\right|}{\widehat{F}_{L_1}^{\alpha_1}\left(Z_{1i}\right)\widehat{F}_{L_2}^{\alpha_2}\left(Z_{2i}\right)\widehat{S}_{R_1}^{\beta_1}\left(Z_{1i}\right)\widehat{S}_{R_2}^{\beta_2}\left(Z_{2i}\right)}+\frac{\left|\widehat{S}_{R_1}\left(Z_{1i}\right)-S_{R_1}\left(Z_{1i}\right)\right|}{\widehat{F}_{L_1}^{\alpha_1}\left(Z_{1i}\right)\widehat{F}_{L_2}^{\alpha_2}\left(Z_{2i}\right)\widehat{S}_{R_1}^{\beta_1}\left(Z_{1i}\right)\widehat{S}_{R_2}^{\beta_2}\left(Z_{2i}\right)}$$

$$\left.+\frac{\left|\widehat{S}_{R_2}\left(Z_{2i}\right)-S_{R_2}\left(Z_{2i}\right)\right|}{\widehat{F}_{L_1}^{\alpha_1}\left(Z_{1i}\right)\widehat{F}_{L_2}^{\alpha_2}\left(Z_{2i}\right)\widehat{S}_{R_1}^{\beta_1}\left(Z_{1i}\right)\widehat{S}_{R_2}^{\beta_2}\left(Z_{2i}\right)}\right],$$

where M is a positive constant. So, to prove the lemma, we have to show as in [28] that for $k\in\{1,2\}$

$$\sup_{t\geq\theta_{k_1}}\frac{F_{L_k}\left(t\right)}{\widehat{F}_{L_k}\left(t\right)}=O_P\left(1\right), \tag{A.12}$$

$$\sup_{t\geq\theta_{k_1}}\frac{\left|\widehat{F}_{L_k}\left(t\right)-F_{L_k}\left(t\right)\right|}{\mathcal{K}_{L_k}^{1/2+\varepsilon}(t)F_{L_k}\left(t\right)}=O_P\left(\frac{1}{\sqrt{n}}\right), \tag{A.13}$$

$$\sup_{t\leq\theta_{k_2}}\frac{S_{R_k}\left(t\right)}{\widehat{S}_{R_k}\left(t\right)}=O_P\left(1\right) \tag{A.14}$$

and

$$\sup_{t\leq Z_{(k_n)}}\frac{\left|\widehat{S}_{R_k}\left(t\right)-S_{R_k}\left(t\right)\right|}{\mathcal{K}_{R_k}^{1/2+\varepsilon}(t)S_{R_k}\left(t\right)}=O_P\left(\frac{1}{\sqrt{n}}\right). \tag{A.15}$$

Relation (A.12) follows from the fact that for $t\geq\theta_{k_1}$

$$\frac{F_{L_k}\left(t\right)}{\widehat{F}_{L_k}\left(t\right)}\leq\frac{1}{\widehat{F}_{L_k}\left(\theta_{k_1}\right)}\leq\frac{2}{F_{L_k}\left(\theta_{k_1}\right)}\ a.s.\text{ for n large enough.}$$

Relation (A.13) can be proved in the same way as in [[18], Theorem 2.1].
Relation (A.14) follows from the fact that for $t\leq\theta_{k_2}$

$$\frac{S_{R_k}\left(t\right)}{\widehat{S}_{R_k}\left(t\right)}\leq\frac{1}{\widehat{S}_{R_k}\left(\theta_{k_2}\right)}\leq\frac{2}{S_{R_k}\left(\theta_{k_2}\right)}\ a.s.\text{ for n large enough.}$$

It remains to deal with relation (A.15). Set

$$\xi_{nk}\left(t\right)=\sqrt{n}\left(\frac{\widehat{S}_{R_k}\left(t\right)-S_{R_k}\left(t\right)}{S_{R_k}\left(t\right)}\right)$$

and

$$h\left(t\right)=\frac{1}{\mathcal{K}_{R_k}^{1/2+\varepsilon}\left(t\right)}.$$

It view of [[33], Theorem 7.3], the process $\xi_{nk}\left(t\right)$ converges weakly to a centered Gaussian process in $l^{\infty}\left([0,\tau]\right)$, for any τ such that $\theta_{k_2}<\tau<T_{Z_k}$. So, relation (A.15) can be proved as in [[18], Theorem 2.1]. In fact, it suffices to prove that for all $\varepsilon>0$

$$\lim_{\tau\uparrow T_{Z_k}} \limsup_{n\to\infty} P\left(\sup_{\tau\le t\le Z_{(kn)}} \left|\int_\tau^t h(s)\, d\xi_{nk}(s)\right| > \varepsilon\right) = 0. \tag{A.16}$$

For that, we set

$$F_{L_k}^*(t) = \frac{1}{n}\sum_{i=1}^n I_{\{L_{ki}\le t\}}$$

and

$$S_{R_k}^*(t) = \prod_{i/Z_{ki}\le t}\left(1 - \frac{I_{\{A_{k_i}=1\}}}{n\left(F_{L_k}^*\left(Z_{ki}^-\right) - \widehat{F}_{Z_k}\left(Z_{ki}^-\right)\right)}\right).$$

We have $\xi_{nk}(t) = \xi_{nk}^*(t) + R_{nk}^*(t)$, where

$$\xi_{nk}^*(t) = \sqrt{n}\left(\frac{S_{R_k}^*(t) - S_{R_k}(t)}{S_{R_k}(t)}\right)$$

and

$$R_{nk}^*(t) = \sqrt{n}\left(\frac{\widehat{S}_{R_k}(t) - S_{R_k}^*(t)}{S_{R_k}(t)}\right).$$

Therefore

$$\begin{aligned} P\left(\sup_{\tau\le t\le Z_{(kn)}} \left|\int_\tau^t h(s)\, d\xi_{nk}(s)\right| > \varepsilon\right) &\le P\left(\sup_{\tau\le t\le Z_{(kn)}} \left|\int_\tau^t h(s)\, d\xi_{nk}^*(s)\right| > \varepsilon/2\right) \\ &\quad + P\left(\sup_{\tau\le t\le Z_{(kn)}} \left|\int_\tau^t h(s)\, dR_{nk}^*(s)\right| > \varepsilon/2\right) \\ &=: P_1 + P_2 \end{aligned} \tag{A.17}$$

We start by dealing with P_1. For that, we need the following lemma. Note $\mathcal{F}_{kt}$ the filtration defined by

$$\begin{aligned} \mathcal{F}_{kt} = \mathcal{N} \vee \sigma\big(\big\{&I_{\{X_{ki}\le s\}}, I_{\{L_{ki}\le s\}}, I_{\{R_{ki}\le s\}}, I_{\{X_{ki}\le s, A_{ki}=1\}}, I_{\{L_{ki}\le s, A_{ki}=1\}}, I_{\{R_{ki}\le s, A_{ki}=1\}}, \\ &0 < s \le t, 1\le i\le n\big\}\big), \end{aligned}$$

where $\mathcal{N}$ is the family of negligible sets. □

Lemma A.5. *We have*

i) $\xi_{nk}^*(t)$ *is an* $\mathcal{F}_{kt}-$*martingale.*

ii) $\forall \beta \in \,]0,1[\,,\ P\left(\sup_{t\le Z_{(kn)}} \frac{S_{R_k}^*(t)}{S_{R_k}(t)} \le \frac{1}{\beta}\right) \ge 1-\beta.$

iii) $\sup_{I_{R_k}\le t\le Z_{(kn)}} \left|\frac{F_{L_k}(t) - F_{Z_k}(t)}{F_{L_k}^*(t^-) - \widehat{F}_{Z_k}(t^-)}\right| = O_P(1).$

Proof of Lemma A.5. *i)* Set

$$\Lambda_{R_k}^*(t) = \int_0^t \frac{dH_{nk}^{(1)}(u)}{F_{L_k}^*(u^-) - \widehat{F}_{Z_k}(u^-)}.$$

In view of [[17], Proposition A.4.1], we have

$$\frac{S^*_{R_k}(t)}{S_{R_k}(t)} = 1 - \int_0^t \frac{S^*_{R_k}(u^-)}{S_{R_k}(u)} d\left(\Lambda^*_{R_k}(u) - \Lambda_{R_k}(u)\right). \tag{A.18}$$

Moreover, consider the $\mathcal{F}_{kt}-$martingale

$$M_k(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[I_{\{Z_{ki} \le t, A_{ki}=1\}} - \int_0^t I_{\{L_{k_i} \le u, Z_{ki} \ge u\}} d\Lambda_{R_k}(u) \right].$$

Since $d\left(\Lambda^*_{R_k}(u) - \Lambda_{R_k}(u)\right) = \frac{\sqrt{n}dM_k(u)}{n\left(F^*_{L_k}(u^-) - \widehat{F}_{Z_k}(u^-)\right)}$, relation (A.18) implies that

$$\frac{S^*_{R_k}(t)}{S_{R_k}(t)} = 1 - \sqrt{n} \int_0^t \frac{S^*_{R_k}(u^-)}{nS_{R_k}(u)\left(F^*_{L_k}(u^-) - \widehat{F}_{Z_k}(u^-)\right)} dM_k(u) \tag{A.19}$$

which implies that

$$\xi^*_{nk}(t) = -\int_0^t \frac{S^*_{R_k}(u^-)}{S_{R_k}(u)\left(F^*_{L_k}(u^-) - \widehat{F}_{Z_k}(u^-)\right)} dM_k(u).$$

Since $\frac{S^*_{R_k}(u^-)}{S_{R_k}(u)\left(F^*_{L_k}(u^-) - \widehat{F}_{Z_k}(u^-)\right)}$ is predictable with respect to $\mathcal{F}_{kt}$, Theorem 1 page 890 of [37] shows that $\xi^*_{nk}(t)$ is an $\mathcal{F}_{kt}-$martingale.

ii) Using [[37], Theorem 1 page 890], relation (A.19) shows that $\frac{S^*_{R_k}(t)}{S_{R_k}(t)}$ is an $\mathcal{F}_{kt}-$martingale. So, the claimed result can be proved in the same way as in [[17], Theorem 3.2.1].

iii) Set $Y_{nk}(t) = \frac{1}{n} \sum_{i=1}^{n} I_{\{X_{ki} \ge t, R_{ki} \ge t\}}$ and $e_{nk}(t) = F^*_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-) - F^*_{L_k}(t^-) Y_{nk}(t)$.
We have for all $t \in \left[I_{R_k}, Z_{(kn)}\right]$

$$F^*_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-) \ge F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t) + e_{nk}(t).$$

So

$$\frac{1}{F^*_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)} \le \frac{1}{F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t) + e_{nk}(t)}. \tag{A.20}$$

Furthermore, we have

$$\begin{aligned} &\sup_{I_{R_k} \le t \le Z_{(kn)}} \left| \frac{1}{F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t) + e_{nk}(t)} - \frac{1}{F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t)} \right| \\ &= \sup_{I_{R_k} \le t \le Z_{(kn)}} \left| \frac{e_{nk}(t)}{\left(F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t) + e_{nk}(t)\right) F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t)} \right|. \end{aligned}$$

Since for $t > T_{L_k}$, we have $F^*_{L_k}(t^-) = 1$, so

$$\begin{aligned} e_{nk}(t) &= 1 - \widehat{F}_{Z_k}(t^-) - Y_{nk}(t) \\ &= \frac{1}{n} \sum_{i=1}^{n} I_{\{X_{k_i} \wedge R_{k_i} \ge t\}} - \frac{1}{n} \sum_{i=1}^{n} I_{\{X_{k_i} \ge t, R_{k_i} \ge t\}} \\ &= 0 \quad a.s. \end{aligned}$$

$\left(I_{\{Z_{k_i} \geq t\}} = I_{\{X_{k_i} \wedge R_{k_i} \geq t\}}\ a.s. \text{ because } L_{ki} < t\ \ a.s. \text{ since } t > T_{L_k}\right)$.
Therefore

$$\sup_{I_{R_k} \leq t \leq Z_{(kn)}} \left| \frac{1}{F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t) + e_{nk}(t)} - \frac{1}{F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t)} \right|$$
$$= \sup_{I_{R_k} \leq t \leq T_{L_k}} \left| \frac{e_{nk}(t)}{F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t) \left(F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t) + e_{nk}(t)\right)} \right|.$$

Moreover, for $I_{R_k} \leq t \leq T_{L_k}$, we have for n large enough

$$\begin{aligned} F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t) &\geq F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}\left(T_{L_k}\right) \\ &\geq \frac{F_{L_k}\left(I_{R_k}\right) S_{X_k}\left(T_{L_k}\right) S_{R_k}\left(T_{L_k}\right)}{2} \\ &=: \alpha \end{aligned}$$

and $F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t) + e_{nk}(t) \geq \alpha + e_{nk}(t) > 0$, for n large enough (since $\sup_{I_{R_k} \leq t \leq T_{L_k}} |e_{nk}(t)| = o_{a.s.}(1)$). Thus

$$\begin{aligned} \sup_{I_{R_k} \leq t \leq Z_{(kn)}} \left| \frac{1}{F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t) + e_{nk}(t)} - \frac{1}{F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t)} \right| &\leq \sup_{I_{R_k} \leq t \leq T_{L_k}} \left| \frac{e_{nk}(t)}{\alpha\left(\alpha + e_{nk}(t)\right)} \right| \\ &\leq \frac{2}{\alpha^2} \sup_{I_{R_k} \leq t \leq T_{L_k}} |e_{nk}(t)| \text{ for } n \text{ large enough} \\ &= o_{a.s.}(1). \end{aligned}$$

So relation (A.20) implies that

$$\frac{1}{F^*_{L_k}\left(t^-\right) - \widehat{F}_{Z_k}\left(t^-\right)} \leq \frac{1}{F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t)} + o_{a.s.}(1)$$

where the $o_{a.s.}(1)$ is uniform on $t \in \left[I_{R_k}, Z_{(kn)}\right]$. So

$$\begin{aligned} \frac{F_{L_k}(t) - F_{Z_k}(t)}{F^*_{L_k}\left(t^-\right) - \widehat{F}_{Z_k}\left(t^-\right)} &\leq \frac{F_{L_k}(t) S_{X_k}(t) S_{R_k}(t)}{F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t)} + o_{a.s.}(1) \\ &\leq \frac{S_{X_k}(t) S_{R_k}(t)}{F^*_{L_k}\left(I^-_{R_k}\right) Y_{nk}(t)} + o_{a.s.}(1). \end{aligned} \tag{A.21}$$

Since

$$\frac{1}{F^*_{L_k}\left(I^-_{R_k}\right)} \xrightarrow{a.s.} \frac{1}{F_{L_k}\left(I_{R_k}\right)}, \quad \text{as } n \to \infty$$

we have

$$\frac{1}{F^*_{L_k}\left(I^-_{R_k}\right)} = O_P(1) \tag{A.22}$$

and

$$\sup_{I_{R_k} \leq t \leq Z_{(kn)}} \left| \frac{S_{X_k}(t) S_{R_k}(t)}{Y_{nk}(t)} \right| = O_P(1)$$

(see [[39], Remark 1 (ii)]). Combining this with (A.21) and (A.22) gives the claimed result. This ends the proof of Lemma A.5.

□

Using this lemma, we can show that

$$\lim_{\tau\uparrow T_{Z_k}} \limsup_{n\to\infty} P\left(\sup_{\tau\le t\le Z_{(kn)}}\left|\int_\tau^t h(s)\,d\xi_{nk}^*(s)\right| > \varepsilon/2\right) = 0 \tag{A.23}$$

in the same way as in [[18], Theorem 2.1], see also [[1], Proposition 3].
Now, we deal with the probability P_2. We have for $\tau \le t \le Z_{(kn)}$

$$\begin{aligned}
\left|\frac{\widehat{S}_{R_k}(t) - S^*_{R_k}(t)}{S_{R_k}(t)}\right| &\le \frac{1}{S_{R_k}(t)} \sum_{i/Z_{ki}\le t}\left|\frac{I_{\{A_{ki}=1\}}}{n}\left[\frac{1}{\widehat{F}_{L_k}(Z_{ki}^-) - \widehat{F}_{Z_k}(Z_{ki}^-)} - \frac{1}{F^*_{L_k}(Z_{ki}^-) - \widehat{F}_{Z_k}(Z_{ki}^-)}\right]\right| \\
&\le \sup_{\tau\le t\le Z_{(kn)}}\left|\frac{\widehat{F}_{L_k}(t^-) - F^*_{L_k}(t^-)}{S_{R_k}(t)\left(\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)\right)\left(F^*_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)\right)}\right| \times \frac{1}{n}\sum_{i=1}^n I_{\{A_{ki}=1\}} \\
&\le \sup_{\tau\le t\le T_{L_k}}\left|\frac{\widehat{F}_{L_k}(t^-) - F^*_{L_k}(t^-)}{S_{R_k}(t)\left(\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)\right)\left(F^*_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)\right)}\right|
\end{aligned}$$

(since for $t > T_{L_k}$, $\widehat{F}_{L_k}(t^-) = F^*_{L_k}(t^-) = 1$).
Therefore

$$\begin{aligned}
\sqrt{n}\sup_{\tau\le t\le Z_{(kn)}}\left|\frac{\widehat{S}_{R_k}(t) - S^*_{R_k}(t)}{S_{R_k}(t)}\right| &\le \frac{1}{S_{R_k}(T_{L_k})} \\
&\times \frac{\sqrt{n}\sup_{\tau\le t\le T_{L_k}}\left|\widehat{F}_{L_k}(t^-) - F^*_{L_k}(t^-)\right|}{\inf_{\tau\le t\le T_{L_k}}\left|\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)\right| \inf_{\tau\le t\le T_{L_k}}\left|F^*_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)\right|}.
\end{aligned} \tag{A.24}$$

Moreover, since $\sqrt{n}\left(\widehat{F}_{L_k} - F_{L_k}\right)$ and $\sqrt{n}\left(F^*_{L_k} - F_{L_k}\right)$ converge weakly in $l^\infty([\tau, T_{L_k}])$, we get

$$\sqrt{n}\sup_{\tau\le t\le T_{L_k}}\left|\widehat{F}_{L_k}(t^-) - F^*_{L_k}(t^-)\right| = O_P(1) \tag{A.25}$$

and we have for all $\tau \le t \le T_{L_k}$

$$\begin{aligned}
&\sup_{\tau\le t\le T_{L_k}}\left|\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-) - (F_{L_k}(t) - F_{Z_k}(t))\right| \\
&\ge F_{L_k}(t) - F_{Z_k}(t) - \left|\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)\right| \\
&\ge F_{L_k}(I_{R_k})\,S_{X_k}(T_{L_k})\,S_{R_k}(T_{L_k}) - \left|\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)\right|.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)\right| &\ge F_{L_k}(I_{R_k})\,S_{X_k}(T_{L_k})\,S_{R_k}(T_{L_k}) \\
&\quad - \sup_{I_{R_k}\le t\le T_{L_k}}\left|\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-) - (F_{L_k}(t) - F_{Z_k}(t))\right|
\end{aligned}$$

and

$$\begin{aligned}
\inf_{\tau\le t\le T_{L_k}}\left|\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-)\right| &\ge F_{L_k}(I_{R_k})\,S_{X_k}(T_{L_k})\,S_{R_k}(T_{L_k}) \\
&\quad - \sup_{I_{R_k}\le t\le T_{L_k}}\left|\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-) - (F_{L_k}(t) - F_{Z_k}(t))\right| \\
&=: \beta - \sup_{I_{R_k}\le t\le T_{L_k}}\left|\widehat{F}_{L_k}(t^-) - \widehat{F}_{Z_k}(t^-) - (F_{L_k}(t) - F_{Z_k}(t))\right|.
\end{aligned}$$

Therefore

$$\frac{1}{\inf\limits_{\tau\le t\le T_{L_k}}\left|\widehat{F}_{L_k}\left(t^-\right)-\widehat{F}_{Z_k}\left(t^-\right)\right|}\le\frac{1}{\beta-\sup\limits_{I_{R_k}\le t\le T_{L_k}}\left|\widehat{F}_{L_k}\left(t^-\right)-\widehat{F}_{Z_k}\left(t^-\right)-\left(F_{L_k}\left(t\right)-F_{Z_k}\left(t\right)\right)\right|}=O_P\left(1\right) \tag{A.26}$$

because

$$\sup_{I_{R_k}\le t\le T_{L_k}}\left|\widehat{F}_{L_k}\left(t^-\right)-\widehat{F}_{Z_k}\left(t^-\right)-\left(F_{L_k}\left(t\right)-F_{Z_k}\left(t\right)\right)\right|\xrightarrow{a.s.}0,\quad\text{as }n\to\infty$$

and with the same manner we can show that

$$\frac{1}{\inf\limits_{\tau\le t\le T_{L_k}}\left|F^*_{L_k}\left(t^-\right)-\widehat{F}_{Z_k}\left(t^-\right)\right|}=O_P\left(1\right).$$

Combining this with (A.24), (A.25) and (A.26), we obtain

$$\sup_{\tau\le t\le Z_{(kn)}}\left|R^*_{nk}\left(t\right)\right|=O_P\left(1\right).$$

Moreover, using an integration by parts we can write

$$\begin{aligned}\left|\int_\tau^t h\left(s\right)dR^*_{nk}\left(s\right)\right|&=\left|h\left(t\right)R^*_{nk}\left(t\right)-h\left(\tau\right)R^*_{nk}\left(\tau\right)-\int_\tau^t R^*_{nk}\left(s\right)dh\left(s\right)\right|\\&\le 2\sup_{\tau\le s\le Z_{(kn)}}\left|h\left(s\right)\right|\sup_{\tau\le s\le Z_{(kn)}}\left|R^*_{nk}\left(s\right)\right|+\sup_{\tau\le s\le Z_{(kn)}}\left|R^*_{nk}\left(s\right)\right|\left|h\left(t\right)-h\left(\tau\right)\right|\\&\le 4\sup_{\tau\le s\le Z_{(kn)}}\left|h\left(s\right)\right|\sup_{\tau\le s\le Z_{(kn)}}\left|R^*_{nk}\left(s\right)\right|.\end{aligned}$$

So

$$P\left(\sup_{\tau\le t\le Z_{(kn)}}\left|\int_\tau^t h\left(s\right)dR^*_{nk}\left(s\right)\right|>\varepsilon/2\right)\le P\left(\sup_{\tau\le t\le T_{Z_k}}\left|h\left(t\right)\right|\sup_{\tau\le t\le Z_{(kn)}}\left|R^*_{nk}\left(t\right)\right|>\varepsilon/8\right). \tag{A.27}$$

It remains to show that

$$\lim_{t\uparrow T_{Z_k}}\limsup_{n\to\infty}P\left(\sup_{\tau\le t\le T_{Z_k}}\left|h\left(t\right)\right|\sup_{\tau\le s\le Z_{(kn)}}\left|R^*_{nk}\left(t\right)\right|>\varepsilon/8\right)=0.$$

This is equivalent to

$$\forall\delta>0,\exists\eta_\delta>0:\left|t-T_{Z_k}\right|<\eta_b\Rightarrow\limsup_{n\to\infty}P\left(\sup_{\tau\le t\le T_{Z_k}}\left|h\left(t\right)\right|\sup_{\tau\le t\le Z_{(kn)}}\left|R^*_{nk}\left(t\right)\right|>\varepsilon/8\right)\le\delta.$$

Let $\delta>0$, since $\sup\limits_{\tau\le t\le Z_{(kn)}}\left|R^*_{nk}\left(t\right)\right|=O_P\left(1\right)$, there exist $b_\delta>0$ and $m_\delta\in\mathbb{N}^*/\forall m\ge m_\delta$

$$\begin{aligned}&P\left(\sup_{\tau\le t\le Z_{(kn)}}\left|R^*_{mk}\left(t\right)\right|>b_\delta\right)<\delta\\&\Rightarrow\sup_{m\ge m_\delta}P\left(\sup_{\tau\le t\le Z_{(kn)}}\left|R^*_{mk}\left(t\right)\right|>b_\delta\right)\le\delta\\&\Rightarrow\limsup_{n\to\infty}P\left(\sup_{\tau\le t\le Z_{(kn)}}\left|R^*_{nk}\left(t\right)\right|>b_\delta\right)\le\delta\end{aligned}$$

and since $\lim\limits_{\tau \uparrow T_{Z_k}} \sup\limits_{\tau \leq t \leq T_{Z_k}} |h(t)| = 0$, there exists $\eta_\delta > 0$ such that

$$|\tau - T_{Z_k}| < \eta_\delta \Rightarrow \sup_{\tau \leq t \leq T_{Z_k}} |h(t)| \leq \frac{\varepsilon}{8 b_\delta}.$$

So, for τ such that $|\tau - T_{Z_k}| < \eta_\delta$, we have

$$P\left(\sup_{\tau \leq t \leq T_{Z_k}} |h(t)| \sup_{\tau \leq t \leq Z_{(kn)}} |R^*_{nk}(t)| > \varepsilon/8\right) \leq P\left(\sup_{\tau \leq t \leq Z_{(kn)}} |R^*_{nk}(t)| > b_\delta\right)$$

$$\Rightarrow \limsup_{n \to \infty} P\left(\sup_{\tau \leq t \leq T_{Z_k}} |h(t)| \sup_{\tau \leq t \leq Z_{(kn)}} |R^*_{nk}(t)| > \varepsilon/8\right) \leq \limsup_{n \to \infty} P\left(\sup_{\tau \leq t \leq Z_{(kn)}} |R^*_{nk}(t)| > b_\delta\right) \leq \delta.$$

Thus

$$\lim_{\tau \uparrow T_{Z_k}} \limsup_{n \to \infty} P\left(\sup_{\tau \leq t \leq T_{Z_k}} |h(t)| \sup_{\tau \leq t \leq Z_{(kn)}} |R^*_{nk}(t)| > \varepsilon/8\right) = 0$$

and relation (A.27) gives

$$\lim_{\tau \uparrow T_{Z_k}} \limsup_{n \to \infty} P\left(\sup_{\tau \leq t \leq Z_{(kn)}} \left|\int_\tau^t h(s)\, dR^*_{nk}(s)\right| > \varepsilon/2\right) = 0.$$

Combining this with (A.17) and (A.23) shows that relation (A.16) is satisfied which ends the proof.

Proof of Theorem 3.3. Using theorem 3.2, this theorem can be proved in the same way as [[19], Theorem 2]. □

Toumi Samia,
Laboratory of Applied Mathematics,
Mohamed Khider University, Biskra,
Algeria.
E-mail address: samia.toumi@univ-biskra.dz

and

Boukeloua Mohamed,
Laboratoire de Génie des Procédés pour le Développement Durable et les Produits de Santé (LGPDDPS),
Ecole Nationale Polytechnique de Constantine,
Algeria
and
Laboratoire de Biostatistique,
Bioinformatique et Méthodologie Mathématique Appliquées aux Sciences de la Santé (BIOSTIM),
Faculté de Médecine, Université Salah Boubnider Constantine 3,
Algeria.
E-mail address: boukeloua.mohamed@gmail.com

and

Idiou Nesrine,
Laboratory of Applied Mathematics,
Mohamed Khider University, Biskra,
Algeria.
E-mail address: nesrine.idiou@univ-biskra.dz

and

Benatia Fatah,
Laboratory of Applied Mathematics,
Mohamed Khider University, Biskra,
Algeria.
E-mail address: fatahbenatia@hotmail.com