On the Sum of the Powers of $A_\alpha$ Eigenvalues of Graphs and $A_\alpha$-energy Like Invariant

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ABSTRACT: For a connected simple graph $G$ with $A_\alpha$ eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ and a real number $\beta$, let $S_\beta^\alpha(G) = \sum_{i=1}^n \rho_i^\beta$ be the sum of the $\beta^{th}$ powers of the $A_\alpha$ eigenvalues of graph $G$. In this paper, we obtain various bounds for the graph invariant $S_\beta^\alpha(G)$ in terms of different graph parameters. As a consequence, we obtain the bounds for the quantity $I E^{A_\alpha}(G) = S_{\frac{1}{2}}^\alpha(G)$, the $A_\alpha$ energy-like invariant of the graph $G$.

Key Words: Adjacency matrix, $A_\alpha$ matrix, Degree regular graph, Signless Laplacian Matrix.

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1. Introduction

Let $G(V, E)$ be a simple graph with $n$ vertices and $m$ edges and having vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. This is referred as $(n, m)$ graph. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, is the neighborhood of $v$. The degree of $v$, denoted by $d_G(v)$ (we simply write $d_v$ if it is clear from the context) is the cardinality of $N(v)$. A graph is called regular if each of its vertices have the same degree. The adjacency matrix $A = (a_{ij})$ of $G$ is a $(0, 1)$-square matrix of order $n$ whose $(i,j)$-entry is equal to 1, if $v_i$ is adjacent to $v_j$ and equal to 0, otherwise. Let $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees $d_i = d_G(v_i), i = 1, 2, \ldots, n$ of graph $G$. The matrices $L(G) = A(G) - D(G)$ and $Q(G) = A(G) + D(G)$ are called the Laplacian and the signless Laplacian matrix, respectively. It is well known that both $L(G)$ and $Q(G)$ are positive semidefinite matrices having real eigenvalues so that their eigenvalues can be ordered as $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ and $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$, respectively.

Nikiforov [10] proposed to study the convex combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ defined by $A_\alpha(G) = \alpha D(G) + (1 - \alpha) A(G), 0 \leq \alpha \leq 1$. It is obvious that $A(G) = A_0(G)$, $D(G) = A_1(G)$ and $2A_{\frac{1}{2}}(G) = D(G) + A(G) = Q(G)$. We further note that $A_\alpha - A_\gamma = (\alpha - \gamma)(D(G) - A(G)) = (\alpha - \gamma)L(G)$. As $A_\alpha(G)$ is a symmetric matrix, for $\alpha \in [\frac{1}{2}, 1]$, clearly $A_\alpha(G)$ is positive semidefinite and so the $A_\alpha$ eigenvalues of $G$ can be taken as $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G)$. In this setup, the matrices $A(G), Q(G)$ and $D(G)$ were seen in a new light and very interesting results were deduced in [3,10,11,14,17].

For a real number $\beta$ ($\beta \neq 0, 1$), Zhou [18] considered the graph invariant $s_\beta(G)$, the sum of $\beta^{th}$ powers of the Laplacian eigenvalues of $G$. In particular, for $\beta = \frac{1}{2}$, $s_{\frac{1}{2}}(G) = \sum_{i=1}^n \sqrt{\mu_i} = LEL(G)$, known as Laplacian-energy-like invariant, was investigated in [9]. Similarly for $\beta = -1$, we have $ns_{-1}(G) = \sum_{i=1}^n \frac{1}{\mu_i} = Kf(G)$, called the Kirchhoff index [4] of the graph $G$. We note that the cases $\beta = 0, 1$ are trivial as $s_0(G) = n - 1$ and $s_1(G) = Tr(L(G)) = 2m$, where $Tr$ is the trace of the matrix. More about LEL($G$) and $Kf(G)$ can be found in [13] and the references therein.

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Akbari et al. [1] introduced the sum of the $\beta^{th}$ powers of the signless Laplacian eigenvalues of $G$ as $s^+_\beta(G) = \sum_{i=1}^{n} q_i^\beta$. Again for $\beta = 0, 1$, we have $s^+_0(G) = n$ and $s^+_1(G) = 2m$. Likewise for $\beta = \frac{1}{2}$, we have $s^+_\frac{1}{2}(G) = \sum_{i=1}^{n} \sqrt{q_i} = IE(G)$, known as incidence energy of the graph $G$.

Motivating the definitions of $s_\beta(G)$ and $s^+_\beta(G)$, we put forward $S^\alpha_\beta(G) = \sum_{i=1}^{n} \rho_i^\beta$, for the sum of the $\beta^{th}$ powers of the $A_\alpha$ eigenvalues of the graph $G$. If $\beta = 0$, we get $S^\alpha_0(G) = n$ and for $\beta = 1$, we have $S^\alpha_1(G) = Tr(A_\alpha(G)) = 2am$. To avoid trivialities, we assume $\beta \neq 0, 1$. In particular for $\beta = \frac{1}{2}$, we obtain $S^\alpha_\frac{1}{2}(G) = \sum_{i=1}^{n} \sqrt{\rho_i} = IE^{A_\alpha}(G)$. This quantity is similar to $LEL(G)$ and $IE(G)$ and is called $A_\alpha$-energy-like invariant.

The first general Zagreb index [7] (also called the general zeroth-order Randić index) of a graph $G$ is denoted by $Z_a(G)$ and is defined as $Z_a(G) = \sum_{i=1}^{n} d_i^a$, where $a$ is any real number other than 0 and 1.

Also, for $a = 2$, we have $Z_2(G) = \sum_{i=1}^{n} d_i^2 = M_1(G)$, which is known as the first Zagreb index [5] of $G$. For concepts and notations not defined here, we refer the reader to any standard text, such as [2,6,15].

The following inequalities play an important role in Sections 2 and 3.

**Lemma 1.1** (Power mean inequality). If $q > p > 0$, and $x_1, x_2, \ldots, x_n$ are non negative real numbers, then

$$
\left( \frac{x_1^p + x_2^p + \cdots + x_n^p}{n} \right)^\frac{1}{p} \geq \left( \frac{x_1^q + x_2^q + \cdots + x_n^q}{n} \right)^\frac{1}{q},
$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

**Lemma 1.2** (Jensen’s inequality). Let $f$ be a convex function on an interval $\mathcal{I}$ and let $x_1, x_2, \ldots, x_n$ be points of $\mathcal{I}$ and let $a_1, a_2, \ldots, a_n$ be real numbers satisfying $\sum_{k=1}^{n} a_k = 1$. Then

$$
f \left( \sum_{k=1}^{n} a_k x_k \right) \leq \sum_{k=1}^{n} a_k f(x_k)
$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

The following lemmas will be used in the sequel.

**Lemma 1.3.** [10,14] Let $G$ be a connected graph of order $n$ and size $m$ having vertex degree sequence $\{d_1, d_2, \ldots, d_n\}$. Then

1. $\sum_{i=1}^{n} \rho_i = 2am$.
2. \(\sum_{i=1}^{n} \rho_i^2 = \alpha^2 Z_2(G) + (1 - \alpha)^2 2m\).
3. \(\sum_{i=1}^{n} s_i^2 = \alpha^2 Z_2(G) + (1 - \alpha)^2 2m - \frac{4\alpha m^2}{n}\).
4. $\rho(G) \geq \frac{2m}{n}$, equality holds if and only if $G$ is degree regular graph.
5. $\rho(G) \geq \sqrt{\frac{Z_2(G)}{n}}$, equality holds if and only if $G$ is degree regular graph.

**Lemma 1.4.** [10] Let $G$ be a connected graph of order $n$ with diameter $D$. If $A_\alpha$ has exactly $t$ distinct eigenvalues, then $D + 1 \leq t$.

**Lemma 1.5.** [10] Let $G$ be a connected graph of order $n$ with $\alpha \in \left[\frac{1}{2}, 1\right]$. Then $A_\alpha$ is a positive semidefinite matrix. If $G$ has no isolated vertices then $A_\alpha$ is positive definite.
From Lemma 1.5, for \( \alpha \in \left[ \frac{1}{2}, 1 \right] \), we see that \( A_{\alpha} \) is a positive semidefinite matrix, so that \( \rho_i(G) \geq 0 \) for \( i = 1, 2, \ldots, n \). From now onwards, we assume that \( \alpha \in \left[ \frac{1}{2}, 1 \right] \) unless otherwise stated.

**Lemma 1.6.** [14] Let \( G \) be a connected graph of order \( n \) and size \( m \), where \( m \geq n \) and let \( G' = G - e \) be a connected graph obtained from \( G \) by deleting an edge. Then \( \rho_i(A_{\alpha}(G)) \geq \rho_i(A_{\alpha}(G')) \) holds for all \( 1 \leq i \leq n \).

**Lemma 1.7.** [10] The \( A_{\alpha} \) eigenvalues of the complete graph \( K_n \) are \( \{n - 1, (\alpha n - 1)^{\lfloor n - 1 \rfloor} \} \), where \( \lfloor j \rfloor \) means the multiplicity of \( \lambda \).

**Lemma 1.8.** [14] Let \( G \) be a connected graph of order \( n \) having vertex degree sequence \( [d_1, d_2, \ldots, d_n] \). Then \( \rho(G) \geq \sqrt{\frac{Z_2(G)}{n}} \geq \frac{2m}{n} \), with equalities if and only if \( G \) is degree regular.

**Lemma 1.9.** [10] Let \( G \) be a graph with maximum degree \( \Delta(G) = \Delta \). Then
\[
\rho(G) \geq \frac{1}{2} \left( \alpha(\Delta + 1) + \sqrt{\alpha^2(\Delta + 1)^2 + 4 \Delta (1 - 2\alpha)} \right).
\]
If \( \alpha \in [0, 1) \) and \( G \) is a connected graph, equality holds if and only if \( G \cong K_{1, \Delta} \).

In Section 2, we obtain upper and lower bounds for \( S_{\beta}^\beta(G) \) in terms of different parameters related to graphs like maximum degree, number of edges, trace of \( A_{\alpha} \), clique number, independence number and other parameters. In Section 3, we obtain bounds for \( IE^{A_{\alpha}}(G) \).

### 2. Bounds for \( S_{\beta}^\beta(G) \)

Let \( G \) be a connected \((n, m)\) graph with \( A_{\alpha} \) eigenvalues \( \rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G) \). For brevity, we use \( \rho_i \) instead of \( \rho_i(G) \). For \( 1 \leq k \leq n - 1 \), let \( M_k = \sum_{i=1}^{k} \rho_i \) and \( m_k = \sum_{i=k+1}^{n} \rho_i \). If \( G \) is connected without isolated vertices and \( \alpha \in \left[ \frac{1}{2}, 1 \right] \), then \( M_k \geq \alpha \sum_{i=1}^{k} 1 = \alpha k \), for \( 1 \leq k \leq n - 1 \), which is a consequence of Schur’s theorem stating that the spectrum of any positive definite symmetric matrix majorizes its main diagonal. This can be further improved as follows:

\[
\frac{M_k}{k} = \frac{\sum_{i=1}^{k} \rho_i}{k} \geq \frac{\sum_{i=k+1}^{n} \rho_i}{n - k} = \frac{2\alpha m - M_k}{n - k}, \quad (2.1)
\]

which after simplification gives \( M_k \geq \frac{2\alpha m k}{n} \). It can be easily verified that equality holds if and only if \( G \cong K_n \). Similarly, we can show that \( m_k \leq \frac{2\alpha m k}{n} \) with equality if and only if \( G \cong K_n \).

Now, we have the following observation.

**Lemma 2.1.** If \( G \) be a connected \((n, m)\) graph having \( m \geq n \) edges, then \( \rho_2(G) = \rho_3(G) = \cdots = \rho_n(G) \) if and only if \( G \cong K_n \).

**Proof.** Suppose \( \rho_2 = \rho_3 = \cdots = \rho_n \). Then \( t = 2 \) and from Lemma 1.4, \( D = 1 \). Conversely, if \( G \cong K_n \). Then \( \rho_2 = \rho_3 = \cdots = \rho_n \) and the result follows.

**Lemma 2.2.** Let \( G \) be a connected \((n, m)\) graph with \( m \geq n \) edges. Then
\[
M_k \geq \frac{2\alpha m k + \{k(n - k)|n(\alpha^2 Z_2(G) + 2m(1 - \alpha)^2) - (2\alpha m)^2]\}^{\frac{1}{2}}
\]
with equality if and only if \( G \cong K_n \).
Proof. Using Cauchy-Schwartz’s inequality and Lemma 1.3, we have

\[(2am - M_k)^2 = \left( \sum_{i=k+1}^{n} \rho_i \right)^2 \leq (n-k) \left( \sum_{i=k+1}^{n} \rho_i^2 \right) = (n-k) \left( \sum_{i=1}^{n} \rho_i^2 - \sum_{i=1}^{k} \rho_i^2 \right)\]

\[= (n-k) \left( \alpha^2 Z_2(G) + (1-\alpha)^2 2m - \sum_{i=1}^{k} \rho_i^2 \right)\]

\[\leq (n-k) \left( \alpha^2 Z_2(G) + (1-\alpha)^2 2m - \frac{M_k^2}{k} \right).\]

After making simplifications, we obtain

\[nM_k^2 - 4amkM_k + 4\alpha^2 m^2 - k(n-k)(\alpha^2 Z_2(G) + 2m(1-\alpha)^2) \leq 0.\]

Hence, it follows that

\[M_k \leq \frac{2amk + \sqrt{k(n-k)|n(\alpha^2 Z_2(G) + 2m(1-\alpha)^2) - 4\alpha^2 m^2|}}{n}\]

which is inequality (2.2).

Assume that equality holds in (2.2). Then all above inequalities must be equalities. So \(\rho_1 = \rho_2 = \cdots = \rho_k\) and \(\rho_{k+1} = \rho_{k+2} = \cdots = \rho_n\), that is, \(G\) has exactly two distinct \(A_\alpha\) eigenvalues. So, by Equation (2.1), \(G \cong K_n\). Similarly it is easy to check equality other way round. \(\square\)

Inequality (2.2) can also be written in terms of the trace of the matrix as

\[M_k \leq \frac{kTr(A_\alpha) + \sqrt{k(n-k)|n(\alpha^2 Z_2(G) + (1-\alpha)^2 Tr(A^2)) - (Tr(A^2))^2|}}{n}.\]

If we proceed similar to Lemma 2.2, we have

\[m_k \geq \frac{2amk + \left\{ k(n-k)|n(\alpha^2 Z_2(G) + 2m(1-\alpha)^2) - (2am)^2| \right\}^{\frac{1}{2}}}{n}\]

with equality if and only if \(G \cong K_n\).

If \(\rho_1\) and \(\rho_n\) are respectively the largest and the smallest \(A_\alpha\) eigenvalues, for \(k = 1\), then Lemmas 2.2 and 2.3 imply that

\[\rho_1 \leq \frac{2am + \left\{ (n-1)|n(\alpha^2 Z_2(G) + 2m(1-\alpha)^2) - (2am)^2| \right\}^{\frac{1}{2}}}{n}\]

and

\[\rho_n \geq \frac{2am + \left\{ (n-1)|n(\alpha^2 Z_2(G) + 2m(1-\alpha)^2) - (2am)^2| \right\}^{\frac{1}{2}}}{n}\]

If \(G - e\) is the graph obtained from \(G\) by deleting the edge \(e\), using Lemma (1.6) and the fact that if \(a \leq b\), then \(a^l \leq b^l\) for each \(l > 0\) and \(a^l \geq b^l\) for each \(l < 0\), we get

\[S_{\beta}^n(G) \geq S_{\beta}^n(G - e), \quad \text{if } \beta > 0\]

\[S_{\beta}^n(G) \leq S_{\beta}^n(G - e), \quad \text{if } \beta < 0.\] (2.4)

As \(G\) is a spanning subgraph of \(K_n\), using (3.4) and Lemma (1.7), we have

\[S_{\beta}^n(G) \leq (n-1)^\beta + (n-1)(an-1)^\beta, \quad \text{if } \beta > 0\]

\[S_{\beta}^n(G) \geq (n-1)^\beta + (n-1)(an-1)^\beta, \quad \text{if } \beta < 0,\]

with equality occurring in both cases if and only if \(G \cong K_n\).
If $G$ is a connected bipartite graph of order $n$ with partite sets of cardinality $a$ and $b$, then $G$ is the spanning subgraph of the complete bipartite graph $K_{a,b}$. For $n \geq 2$ and $m \geq n$, we have

$$S_\beta^\alpha(G) \leq x_1^\beta + x_2^\beta + (b-1)(aa)^\beta + (a-1)(ab)^\beta, \quad \text{if } \beta > 0$$

$$S_\beta^\alpha(G) \geq x_1^\beta + x_2^\beta + (b-1)(aa)^\beta + (a-1)(ab)^\beta, \quad \text{if } \beta < 0,$$

where $x_1 = \frac{1}{2}(an + \sqrt{(an)^2 + 4ab(1-2\alpha)})$ and $x_2 = \frac{1}{2}(an - \sqrt{(an)^2 + 4ab(1-2\alpha)})$, equality occurring in both cases if and only if $G \cong K_{a,b}$.

A complete split graph, denoted by $CS_{n-k,k}$, is the graph consisting of an independent set on $k$ vertices and a clique on $n-k$ vertices, such that each vertex of the clique is connected to every vertex of the independent set. It is well known that $CS_{n-k,k} = K_{n-k} \vee K_k$. Using this information in Proposition 37 of [10], we can find $A_\alpha$ spectrum of $CS_{n-k,k}$.

For $\alpha \in [0,1]$, the eigenvalues of $A_\alpha(CS_{n-k,k})$ are

$$\left\{ \frac{n-k-1+\alpha n \pm \sqrt{\theta}}{2}, \(\alpha(n-k)\)_{[k-1]}^{[\alpha n-\alpha+1]} \right\},$$

where $\theta = k^2(4\alpha - 3) + k(2n + 2 - 2an - 4\alpha) + n(\alpha - 1)(n\alpha - \alpha + 2) + 1$.

In case $G$ is a connected graph on $n \geq 2$ vertices having independence number $k$, then

$$S_\beta^\alpha(G) \leq x_1^\beta + x_2^\beta + (k-1)(an - b\alpha)^\beta + (n-k-1)(an - 1)^\beta, \quad \text{if } \beta > 0$$

$$S_\beta^\alpha(G) \geq x_1^\beta + x_2^\beta + (k-1)(an - b\alpha)^\beta + (n-k-1)(an - 1)^\beta, \quad \text{if } \beta < 0,$$

where

$$x_1 = \frac{1}{2} \left[ n-k-1 + an + \left\{ k^2(4\alpha - 3) + k(2n + 2 - 2an - 4\alpha) + n(\alpha - 1)(n\alpha - \alpha + 2) + 1 \right\}^{\frac{1}{2}} \right]$$

and

$$x_2 = \frac{1}{2} \left[ n-k-1 + an - \left\{ k^2(4\alpha - 3) + k(2n + 2 - 2an - 4\alpha) + n(\alpha - 1)(n\alpha - \alpha + 2) + 1 \right\}^{\frac{1}{2}} \right],$$

equality occurring in both cases if and only if $G \cong CS_{n-k,k}$.

Further, if $G$ is a degree regular graph on $n \geq 3$ vertices, then

$$S_\beta^\alpha(C_n) \leq S_\beta^\alpha(G) \leq (n-1)^\beta + (n-1)(an - 1)^\beta, \quad \text{if } \beta > 0$$

$$S_\beta^\alpha(C_n) \geq S_\beta^\alpha(G) \geq (n-1)^\beta + (n-1)(an - 1)^\beta, \quad \text{if } \beta < 0,$$

equality holds on the right if and only if $G \cong K_n$ and equality occurs on the left if and only if $G \cong C_n$.

**Theorem 2.3.** Let $G$ be a connected graph of order $n \geq 2$.

(i) If $\beta < 0$ or $\beta > 1$, then

$$S_\beta^\alpha(G) \geq \left( \frac{2m}{n} \right)^\beta + \frac{(2m(an-1))^{\beta}}{n^\beta(an-1)^{\beta-1}},$$

with equality if and only if $G \cong K_n$.

(ii) If $0 < \beta < 1$, then

$$S_\beta^\alpha(G) \leq \left( \frac{2m}{n} \right)^\beta + \frac{(2m(an-1))^{\beta}}{n^\beta(an-1)^{\beta-1}},$$

with equality if and only if $G \cong K_n$.

**Proof.** For $\beta \neq 0,1$ and $x > 0$, we see that $x^\beta$ is concave up when $\beta < 0$ or $\beta > 1$. Thus, by Jensen’s inequality, we have

$$\left( \sum_{i=2}^{n} \frac{1}{n-1} \rho_i \right)^\beta \leq \sum_{i=2}^{n} \frac{1}{n-1} \rho_i^\beta,$$
which implies that \( \sum_{i=2}^{n} \rho_i^\beta \geq \frac{1}{(n-1)^\beta} \left( \sum_{i=2}^{n} \rho_i \right)^\beta \) with equality if and only if \( \rho_2 = \rho_3 = \cdots = \rho_n \). Now, using this observation in the definition of \( S^\beta_\beta(G) \), we have

\[
S^\beta_\beta(G) \geq \rho_1^\beta + \frac{1}{(n-1)^\beta} \left( \sum_{i=2}^{n} \rho_i \right)^\beta = \rho_1^\beta + \frac{(2\alpha m - \rho_1^\beta)}{(n-1)^\beta-1}.
\]

Let \( f(x) = x^\beta + \frac{(2\alpha m - x^\beta)}{(n-1)^\beta-1} \). By solving \( f'(x) \geq 0 \), we see that \( f(x) \) is increasing for \( x \geq \frac{2\alpha m}{n} \). By Lemma 1.3, we have \( \rho_1 \geq \frac{2m}{n} \geq \frac{2\alpha m}{n} \) and thus

\[
S^\beta_\beta(G) \geq f \left( \frac{2m}{n} \right) = \left( \frac{2m}{n} \right)^\beta + \frac{(2m(\alpha n - 1))^\beta}{n^{\beta-1}(n-1)^{\beta-1}},
\]

with equality if and only if \( \rho_2 = \rho_3 = \cdots = \rho_n \) and \( \rho_1 = \frac{2m}{n} \). Therefore, \( G \) has exactly two distinct \( \lambda_\alpha \) eigenvalues and by Lemma 2.1, \( G \) is the complete graph \( K_n \), proving part (i).

(ii) Suppose that \( 0 < \beta < 1 \). Then, clearly \( x^\beta \) is concave down when \( x > 0 \) or \( 0 < \beta < 1 \). So,

\[
\left( \sum_{i=2}^{n} \frac{1}{n-1} \rho_i \right)^\beta \geq \sum_{i=2}^{n} \frac{1}{n-1} \rho_i^\beta,
\]

with equality if and only if \( \rho_2 = \rho_3 = \cdots = \rho_n \) and \( f(x) \) is decreasing for \( x \geq \frac{2\alpha m}{n} \). Now proceeding as in part (i), we obtain the required result. \( \square \)

Using similar arguments as in Theorem 2.3 and Lemma 1.8, we have the following.

(i) If \( \beta < 0 \) or \( \beta > 1 \), then

\[
S^\alpha_\beta(G) \geq \left( \frac{Z_2(G)}{n} \right)^\beta + \frac{(2m\alpha \sqrt{n} - Z_2(G))^\beta}{n^{\beta-1}(n-1)^{\beta-1}},
\]

with equality if and only if \( G \cong K_n \).

(ii) If \( 0 < \beta < 1 \), then

\[
S^\alpha_\beta(G) \leq \left( \frac{Z_2(G)}{n} \right)^\beta + \frac{(2m\alpha \sqrt{n} - Z_2(G))^\beta}{n^{\beta-1}(n-1)^{\beta-1}},
\]

with equality if and only if \( G \cong K_n \).

**Theorem 2.4.** Let \( G \) be a graph of order \( n \geq 2 \) and \( 1 \leq k \leq n-1 \) be a positive integer.

(i) If \( 0 < \beta < 1 \), then

\[
S^\alpha_\beta(G) \leq k^{1-\beta} \left( \frac{2\alpha m k}{n} \right)^\beta + (n-k)^{1-\beta} \left( \frac{2\alpha m \left( \frac{n-k}{n} \right)}{n} \right)^\beta,
\]

with equality if and only if \( G \cong K_1 \).

(ii) If \( \beta > 1 \), then

\[
S^\alpha_\beta(G) \geq k^{1-\beta} \left( \frac{2\alpha m k}{n} \right)^\beta + (n-k)^{1-\beta} \left( \frac{2\alpha m \left( \frac{n-k}{n} \right)}{n} \right)^\beta,
\]
with equality if and only if $G \cong K_1$.

(iii) If $\beta < 0$, then

$$S_\beta^0(G) \leq k^{1-\beta} \left( \frac{2\alpha mk + \sqrt{\theta}}{n} \right)^\beta + (n-k)^\beta \left( \frac{2\alpha mk - \sqrt{\theta}}{n} \right)^\beta,$$

where $\theta = k(n-k)(n(\alpha^2 Z_2(G) + 2(1-\alpha)^2 m) - (2\alpha m)^2)$.

**Proof.** By power mean inequality with $0 < \beta < 1$, we have

$$\left( \frac{\sum_{i=1}^k \rho_i^\beta}{k} \right)^\frac{1}{\beta} \leq \frac{M_k}{k},$$

that is, $\sum_{i=1}^k \rho_i^\beta \leq k^{1-\beta} M_k^\beta$ with equality if and only if $\rho_1 = \rho_2 = \cdots = \rho_k$.

Similarly, $\sum_{i=k+1}^n \rho_i^\beta \leq (n-k)^{1-\beta} (2\alpha m - M_k)^\beta$, with equality if and only if $\rho_{k+1} = \rho_{k+2} = \cdots = \rho_n$.

Thus, by the definition of $S_\beta^0(G)$, we have

$$S_\beta^0(G) = \sum_{i=1}^k \rho_i^\beta + \sum_{i=k+1}^n \rho_i^\beta \leq k^{1-\beta} M_k^\beta + (n-k)^{1-\beta} (2\alpha m - M_k)^\beta.$$

Consider the function

$$f(x) = k^{1-\beta} x^\beta + (n-k)^{1-\beta} (2\alpha m - x)^\beta, \quad x \geq \frac{2\alpha mk}{n}.$$

We see that

$$f'(x) = \beta \left( \frac{x}{k} \right)^{\beta-1} - \left( \frac{2\alpha m - x}{n-k} \right)^{\beta-1} \leq 0$$

provided $0 < \beta < 1$ and $x \geq \frac{2\alpha mk}{k}$. Thus $f(x)$ is a decreasing function on $x \geq \frac{2\alpha mk}{k}$. Therefore, by equation (2.1), $M_k \geq \frac{2\alpha mk}{n}$ and we have

$$S_\beta^0(G) = f(M_k) \leq f \left( \frac{2\alpha mk}{n} \right) = k^{1-\beta} \left( \frac{2\alpha mk}{n} \right)^\beta + (n-k) \left( \frac{2\alpha m - 2\alpha mk}{n} \right)^\beta,$$

proving part (i).

Suppose equality holds, that is, $\rho_1 = \rho_2 = \cdots = \rho_k, \rho_{k+1} = \rho_{k+2} = \cdots = \rho_n$ and $M_k = \frac{2\alpha mk}{n}$. From this, we have $\rho_1 = \rho_2 = \cdots = \rho_n = \frac{2\alpha m}{n}$, which happens if $G \cong K_1$. Conversely, we can easily verify that equality occurs if $G \cong K_1$.

(ii) For $\beta > 1$, using power mean inequality as in part (i), we obtain

$$S_\beta^0(G) \geq k^{1-\beta} M_k^\beta + (n-k)^{1-\beta} (2\alpha m - M_k)^\beta.$$

Also, $f(x) = k^{1-\beta} x^\beta + (n-k)^{1-\beta} (2\alpha m - x)^\beta$ is an increasing function on $x \geq \frac{2\alpha mk}{n}$ for $\beta > 1$. Now proceeding similarly as in (i) we can establish (ii). Also, the equality can be discussed similar to (i).

(iii) We note that $f(x) = k^{1-\beta} x^\beta + (n-k)^{1-\beta} (2\alpha m - x)^\beta$ is an increasing function on $x \geq \frac{2\alpha mk}{n}$ as $\beta < 0$. From Equation (2.1) and Lemma 2.2, we have

$$\frac{2\alpha mk}{n} \leq x \leq \frac{2\alpha mk + \sqrt{\theta}}{n},$$
where $\theta = k(n - k)(n(\alpha^2 Z_2(G) + 2(1 - \alpha)^2m) - (2\alpha m)^2)$. Hence
\[
S_\beta^\alpha(G) \leq f \left( \frac{2\alpha m k + \sqrt{\theta}}{n} \right) = k^{1 - \beta} \left( \frac{2\alpha m k + \sqrt{\theta}}{n} \right)^\beta + (n - k)^\beta \left( \frac{2\alpha m k - \sqrt{\theta}}{n} \right)^\beta.
\]

Hence
\[
S_\alpha^\beta(G) = \rho_{1}^\beta + (n - 1)D_{n - 1}^{\beta} \left( \frac{2\alpha m k}{n} \right)^{\beta - \frac{\beta}{n - 1}}.
\]

For a connected graph $G$ of order $n \geq 3$, let $D = \prod_{i=1}^{n} \rho_i$, where $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ are the eigenvalues of $A_\alpha$.

**Theorem 2.5.** Let $G$ be a connected $(n, m)$ graph with $n \geq 3$. If $\beta < 0$ or $\beta > 1$, then
\[
S_\beta^\alpha(G) \geq \left( \frac{2m}{n} \right)^\beta + (n - 1)D^{\beta-1} \left( \frac{2m}{n} \right)^{\beta - \frac{\beta}{n - 1}},
\]
with equality if and only if $G \cong K_n$.

**Proof.** From the definition of $S_\beta^\alpha(G)$, we have $S_\beta^\alpha(G) = \rho_{1}^\beta + \sum_{i=2}^{n} \rho_i^\beta$. Applying arithmetic-geometric mean inequality to the second term of the R.H.S, we have
\[
S_\beta^\alpha(G) \geq \rho_{1}^\beta + (n - 1)\left( \prod_{i=2}^{n} \rho_i^\beta \right)^{\frac{1}{n - 1}} = \rho_{1}^\beta + (n - 1) \left( \frac{D}{\rho_1} \right)^{\beta - \frac{\beta}{n - 1}},
\]
with equality if and only if $\rho_2 = \rho_3 = \cdots = \rho_n$. Consider the function
\[
f(x) = x^\beta + (n - 1)D^{\beta-1}x^{\beta - \frac{\beta}{n - 1}}.
\]
After differentiation, we have
\[
f'(x) = \beta x^{\beta - 1} \left( \frac{n\beta}{x^{n - 1} - Dn^{n - 1}} - \frac{\beta}{n - 1} \right).
\]
For $\beta < 0$ or $\beta > 1$, we can easily verify that $f(x)$ is an increasing function for $x \geq D^{\frac{1}{n - 1}}$. Therefore, by Lemma 1.3 and using arithmetic-geometric inequality, we have
\[
\rho_1 \geq \frac{2m}{n} \geq \frac{2\alpha m}{n} = \frac{\sum_{i=1}^{n} \rho_i}{n} \geq \left( \prod_{i=1}^{n} \right)^{\frac{1}{n}} \frac{1}{n} = \frac{D}{n}.
\]
So, this implies that
\[
S_\beta^\alpha(G) \geq f \left( \frac{2m}{n} \right) = \left( \frac{2m}{n} \right)^\beta + (n - 1)D^{\beta-1} \left( \frac{2m}{n} \right)^{\beta - \frac{\beta}{n - 1}}\frac{1}{n - 1}.
\]
Equality occurs if and only if $\rho_1 = \frac{2m}{n}$ and $\rho_2 = \rho_3 = \cdots = \rho_n$. That is, if and only if $G$ is degree regular with two distinct $A_\alpha$ eigenvalues. So, by Lemma 2.1, $G \cong K_n$. \qed
Theorem 2.6. Let \( G \) be a graph of order \( n \geq 2 \) and \( 1 \leq k \leq n - 1 \) be a positive integer.

(i) If \( \beta < 0 \), \( 0 < \beta < 1 \), then

\[
S_\beta^\alpha(G) \geq \frac{(2\alpha m)^{2-\beta}}{(\alpha^2 Z_2(G) + (1-\alpha)^2 2\alpha m)^{1-\beta}}.
\]

(ii) If \( 1 < \beta \leq 2, \beta > 2 \), then

\[
S_\beta^\alpha(G) \leq \frac{(2\alpha m)^{2-\beta}}{(\alpha^2 Z_2(G) + (1-\alpha)^2 2\alpha m)^{1-\beta}}.
\]

Proof. Let \( a_1, a_2, \ldots, a_n \) be positive real numbers and let \( k \) be a real number with \( k \neq 0, \frac{1}{2}, 1 \). It is clear that, \( k < 0 \) or \( k > 0 \), so that \( \frac{2k-1}{k} > 0 \). By Hölder’s inequality, we have

\[
\sum_{i=1}^{n} a_i^k = \sum_{i=1}^{n} a_i^{\frac{k}{2k-1}} a_i^{\frac{k(2k-1)}{2k-1}} \leq \left( \sum_{i=1}^{n} a_i^{\frac{k}{2k-1}} \right)^{\frac{k}{2k-1}} \left( \sum_{i=1}^{n} a_i^{\frac{k(2k-1)}{2k-1}} \right)^{\frac{2k-1}{2k-1}},
\]

which implies that

\[
\sum_{i=1}^{n} a_i \geq \frac{\left( \sum_{i=1}^{n} a_i^k \right)^{\frac{2k-1}{k}}}{\left( \sum_{i=1}^{n} a_i^2 \right)^{\frac{k}{2k-1}}}.
\]

with equality if and only if \( a_1 = a_2 = \ldots = a_n \). Now, letting \( a = \rho_i \) and \( k = \frac{1}{\alpha} \), it implies that

\[
S_\beta^\alpha(G) = \sum_{i=1}^{n} \rho_i^\beta \geq \left( \frac{\sum_{i=1}^{n} \rho_i^k}{\sum_{i=1}^{n} \rho_i^2} \right)^{2-\beta} = \frac{(2\alpha m)^{2-\beta}}{(\alpha^2 Z_2(G) + (1-\alpha)^2 2\alpha m)^{1-\beta}},
\]

for each \( \beta < 0 \) or \( 0 < \beta < 1 \). Similarly, if \( 1 < \beta < 2 \) or \( \beta > 2 \), then \( \frac{1}{2} < k < 1 \) or \( 0 < k < \frac{1}{2} \). Taking \( p = \frac{2k-1}{k}, q = \frac{2k-1}{2k-1} \) and noting that \( p > 0, q < 0 \) if \( \frac{1}{2} < k < 1 \); and \( p < 0, q > 0 \) if \( 0 < k < \frac{1}{2} \). In each of these cases Hölder’s inequality gets reversed and the second part follows. \( \square \)

3. Bounds for \( IE^{A_\alpha} \) energy-like invariant

The graph invariant \( S_\beta^\alpha(G) = \sum_{i=1}^{n} \sqrt{\rho_i} = IE^{A_\alpha}(G) \) is called \( A_\alpha \)-energy-like invariant. From Theorem 2.3, we observe that

\[
IE^{A_\alpha}(G) \leq \sqrt{\frac{2m}{n}} + \sqrt{\frac{2m(\alpha n - 1)(n-1)}{n}},
\]

with equality if and only if \( G \cong K_n \).

Also, we have

\[
IE^{A_\alpha}(G) \leq \left( \frac{Z_2(G)}{n} \right)^{\frac{1}{2}} + \sqrt{\frac{(2\alpha \sqrt{n} - Z_2(G))(n-1)}{n}},
\]

with equality if and only if \( G \cong K_n \).
From Theorem 2.4 part (i), we have

\[ IE^{A_{\alpha}}(G) \leq (\sqrt{k} + n - k) \sqrt{\left( \frac{2\alpha m}{n} \right)}, \]

with equality if and only if \( G \cong K_1 \).

If \( G - e \) is the connected graph obtained from \( G \) by the deletion of an edge \( e \), then

\[ IE^{A_{\alpha}}(G) \geq IE^{A_{\alpha}}(G - e). \]

Further, we have

\[ IE^{A_{\alpha}}(G) \leq \sqrt{(n - 1) + (n - 1)\alpha n - 1}, \]

with equality occurring in both cases if and only if \( G \cong K_n \).

Also

\[ IE^{A_{\alpha}}(G) \leq \sqrt{x_1 + x_2} + (b - 1) \sqrt{(aa) + (a - 1)} \sqrt{(ab)}, \]

where \( x_1 = \frac{1}{2}(n - 1) + \alpha n + \left\{ k^2(4\alpha - 3) + k(2n - 2\alpha n - 4\alpha) + n(\alpha - 1)(\alpha n - \alpha + 2) + 1 \right\} \frac{1}{2} \) and \( x_2 = \frac{1}{2}(n - 1) + \alpha n - \left\{ k^2(4\alpha - 3) + k(2n - 2\alpha n - 4\alpha) + n(\alpha - 1)(\alpha n - \alpha + 2) + 1 \right\} \frac{1}{2} \), equality occurring in both cases if and only if \( G \cong K_{n,b} \).

If \( G \) has independence number \( k \), then

\[ IE^{A_{\alpha}}(G) \leq \sqrt{x_1 + x_2} + (k - 1) \sqrt{(an - ak) + (n - k - 1) (an - 1)}, \]

equality occurs if and only if \( G \cong CS_{n-k,k} \).

From Lemma 1.9, if \( B = \frac{1}{2} \left( \alpha (\Delta + 1) + \sqrt{\alpha^2 (\Delta + 1)^2 + 4 \Delta (1 - 2\alpha)} \right) \), then we can easily see that \( \rho(G) \geq B \geq \frac{2\alpha m}{n} \). From second inequality of (2.4), it follows that

\[ IE^{A_{\alpha}}(G) \leq \sqrt{B} + \sqrt{(n - 1)(2\alpha m - B)}, \]

where equality holds if and only if \( G \cong K_{1,\Delta} \).

**Theorem 3.1.** Let \( G \) be a connected graph \((n, m)\) graph, where \( n \geq 2 \). Then

\[ IE^{A_{\alpha}}(G) \leq \left\{ 2\alpha m + (n - 1) \left( (an - 1)(n - 2\alpha) + n(n - 1) \sqrt{(an - 1)} \right) \right\} \frac{1}{2} \]

where equality holds if and only if \( G \cong K_n \).

**Proof.** Let \( G \) be a connected graph of order \( n \geq 2 \) having \( A_{\alpha} \) eigenvalues \( \rho_1, \rho_2, \ldots, \rho_n \). Now

\[ (IE^{A_{\alpha}}(G))^2 = \left( \sum_{i=1}^{n} \sqrt{\rho_i} \right)^2 = \sum_{i=1}^{n} \rho_i + 2 \sum_{i \neq j} \sqrt{\rho_i} \sqrt{\rho_j}. \]  

(3.1)

As we know \( G \) is a connected spanning subgraph of \( K_n \), thus by Lemma 1.6 and noting that \( \alpha \) lies in \([\frac{1}{2}, 1]\), we have

\[ \rho_1(G) \leq \rho_1(K_n) = n - 1, \quad \rho_i(G) \leq \rho_i(K_n) = \alpha n - 1, \quad i = 2, 3, \ldots, n. \]
Evaluating the second term of (3.1), we have
\[
\sum_{i \neq j} \sqrt{\rho_i \rho_j} = \sqrt{\rho_1 (\sqrt{\rho_2} + \sqrt{\rho_3} + \cdots + \sqrt{\rho_n})} + \sqrt{\rho_2 (\sqrt{\rho_3} + \sqrt{\rho_4} + \cdots + \sqrt{\rho_n})} + \cdots + \sqrt{\rho_{n-1} \sqrt{\rho_n}} \\
\leq (n-1)(\sqrt{(n-1)(\alpha n - 1)}) + (n-2)(\alpha n - 1) + \cdots + (\alpha n - 1) \\
= (n-1)\sqrt{(n-1)(\alpha n - 1)} + (\alpha n - 1) \left(\frac{(n-1)(n-2)}{2}\right).
\]

Hence, from equation (3.1), we obtain
\[
IE^{A_\alpha}(G) \leq \left\{ 2\alpha m + (n-1) \left( (\alpha n - 1)(n-2) + 2(n-1)\sqrt{(n-1)(\alpha n - 1)} \right) \right\}^{\frac{1}{2}}.
\]

Equality occurs if and only if \(\rho_1(G) = \rho_1(K_n) = n-1\) and \(\rho_i(G) = \rho_i(K_n) = \alpha n - 1\) for \(i = 2, 3, \ldots, n\). That is, if and only if \(G \cong K_n\). □

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