Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) v. ???? (??) : 1–11.
ISSN-0037-8712 in press
doi:10.5269/bspm.62933

Common Fuzzy Fixed Point Results for F-Contractive Mappings with Applications

Durdana Lateef

abstract: The aim of this article is to establish some common fixed point theorems for α-fuzzy mappings under F-contraction in the framework of complete metric spaces. To extend and improve some well-known results of literature, new results for multivalued mappings are obtained as application of established results. We have illustrated an appropriate example to rationalize the notions and outcomes. Also we investigated the solution of the domain of words as application of our results to theoretical computer science.

Key Words: Fixed point, F-contractions, α-fuzzy mappings, multivalued mappings.

Contents

1. Introduction and Preliminaries

Heilpern [13] used the concept of fuzzy set to introduce a family of fuzzy mappings, which is an extension and generalization of the multivalued mapping in 1981. He established a fixed point result for fuzzy mapping in metric linear space in 1981. It is important to note that the result proved by Heilpern [13] is an extension of the Nadler fixed point theorem from multivalued mapping to fuzzy mapping. Moreover, we shall use the following notations which have been recorded from [1,5,6,8]:

A fuzzy set in $\mho$ is a function with domain $\mho$ and values belongs to $[0,1]$. If μ is a fuzzy set and $\eth \in \mho$, then the function values $\mu(\eth)$ is alleged t be the grade of membership of $\eth$ in μ. The α -level set of μ is denoted by $[\mu]_\alpha$ and is defined as follows:

$$[\mu]_\alpha = \{\eth : \mu(\eth) \geq \alpha\} \text{ if } \alpha \in (0,1],$$

$$[\mu]_0 = \overline{\{\eth : \mu(\eth) > 0\}}.$$

Let $F(\mho)$ be the class of all fuzzy sets in a metric space $\mho$. For $\mu_1, \mu_2 \in F(\mho)$, $\mu_1 \subset \mu_2$ means $\mu_1(\eth) \leq \mu_2(\eth)$ for each $\eth \in \mho$. We denote the fuzzy set $\chi_{\{\eth\}}$ by $\{\eth\}$ unless and until it is stated, where $\chi_{\{\eth\}}$ is the characteristic function of the crisp set μ_1. Let $\mho_1$ be an arbitrary set, $\mho_2$ be a metric space. A mapping $\Game$ is called fuzzy mapping if $\Game$ is a mapping from $\mho_1$ into $F(\mho_2)$. A fuzzy mapping $\Game$ is a fuzzy subset on $\mho_1 \times \mho_2$ with membership function $\Game(\eth)(\beth)$. The function $\Game(\eth)(\beth)$ is the grade of membership of $\beth$ in $\Game(\eth)$.

Definition 1.1. *[6] Let $\Game_1, \Game_2 : \mho \to F(\mho)$. A point $\eth^* \in \mho$ is alleged to be a common α- fuzzy fixed point of $\Game_1$ and $\Game_2$ if $\exists\ \alpha \in [0,1]$ such that $\eth^* \in [\Game_1\eth^*]_\alpha \cap [\Game_2\eth^*]_\alpha$.*

Now, following the lines in [12], we denote by the set of all continuous mappings $\sigma : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ satisfying the following conditions:

2010 *Mathematics Subject Classification:* 47H10, 46S40, 54H25.
Submitted March 17, 2022. Published November 09, 2022

(ρ_1) $\sigma(1,1,1,2,0), \sigma(1,1,1,0,2), \sigma(1,1,1,1,1) \in (0,1]$,

(ρ_2) for all $(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) \in (\mathbb{R}^+)^5$ and $\alpha \geq 0$, we have

$$\sigma(\alpha\eth_1, \alpha\eth_2, \alpha\eth_3, \alpha\eth_4, \alpha\eth_5) \leq \alpha\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5);$$

(ρ_3) for $\eth_i, \beth_i \in \mathbb{R}^+$, $\eth_i \leq \beth_i$, $i = 1, ..., 5$, we have

$$\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) \leq \sigma(\beth_1, \beth_2, \beth_3, \beth_4, \beth_5)$$

and if $\eth_i, \beth_i \in \mathbb{R}^+$, $i = 1, ..., 4$, then $\sigma(\eth_1, \eth_2, \eth_3, \eth_4, 0) \leq \sigma(\beth_1, \beth_2, \beth_3, \beth_4, 0)$ and $\sigma(\eth_1, \eth_2, \eth_3, 0, \eth_4) \leq \sigma(\beth_1, \beth_2, \beth_3, 0, \beth_4)$.

Example 1.2. *The following functions $\sigma : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ are the elements of :*

(i) $\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) = \eth_1$.
(ii) $\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) = \eth_2 + \eth_3$.
(iii) $\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) = \eth_4 + \eth_5$.
(iv) $\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) = \daleth_1\eth_1 + \daleth_2\eth_2 + \daleth_3\eth_3$, where $\daleth_1,\ \daleth_2,\ \daleth_3 \in [0, +\infty)$ such that $\daleth_1 + \daleth_2 + \daleth_3 \leq 1$.
(v) $\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) = \daleth_1\eth_1 + \daleth_2\eth_2 + \daleth_3\eth_3 + \daleth_4\eth_4 + \daleth_5\eth_5$, where $\daleth_1,\ \daleth_2,\ \daleth_3,\ \daleth_4,\ \daleth_5 \in [0, +\infty)$ such that $\daleth_1 + \daleth_2 + \daleth_3 + \daleth_4 + 2\daleth_5 \leq 1$.
(vi) $\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) = \daleth_1\eth_1 + \daleth_5\eth_5$, where $\daleth_1 \in [0, 1)$ and $\daleth_5 \geq 0$.
(vii) $\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) = \max\left\{\eth_1, \eth_2, \eth_3, \frac{\eth_4+\eth_5}{2}\right\}$.
(viii) $\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) = \max\left\{\eth_1, \eth_2, \eth_3, \eth_4, \eth_5\right\}$.
(ix) $\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) = \max\left\{\eth_1, \frac{\eth_2+\eth_3}{2}, \frac{\eth_4+\eth_5}{2}\right\}$.

Lemma 1.3. *(see. [18])If $\sigma \in$ and $\eth_1, \eth_2 \in \mathbb{R}^+$ are such that*

$$\eth_1 < \max\left\{\begin{array}{c} \sigma\left(\eth_2, \eth_2, \eth_1, \eth_2 + \eth_1, 0\right), \sigma\left(\eth_2, \eth_2, \eth_1, 0, \eth_2 + \eth_1\right), \\ \sigma\left(\eth_2, \eth_1, \eth_2, \eth_2 + \eth_1, 0\right), \sigma\left(\eth_2, \eth_1, \eth_2, 0, \eth_2 + \eth_1\right) \end{array}\right\},$$

then $\eth_1 < \eth_2$.

In 2012, Wardowski [23] commenced a new type of contractions which is alleged to be an F-contraction which is associated with following functions.

Definition 1.4. *Let $F : (0, \infty) \to \mathbb{R}$ be a function satisfying*

(F_1) $F(\eth_1) < F(\eth_2)$ for $\eth_1 < \eth_2$,

(F_2) $\forall\{\eth_j\} \subseteq R^+$, $\lim_{j\to\infty} \eth_j = 0 \Leftrightarrow \lim_{j\to\infty} F(\eth_j) = -\infty$;

(F_3) $\exists\ 0 < r < 1$ such that $\lim_{\eth\to0^+} \eth^r F(\eth) = 0$.

Consistent with Wardowski [23], we designate by $\digamma$ the class of $F : (0, \infty) \to \mathbb{R}$ satisfying F_1, $F2$ and F_3.

Definition 1.5. *[23] Let $(\mho, \wp)$ be a metric space (MS). A mapping $\backepsilon : \mho \to \mho$ is alleged to be an $F-$contraction if $\exists\ \tau > 0$ and $F \in \digamma$ such that*

$$\wp(\backepsilon\eth, \backepsilon\beth) > 0 \Longrightarrow \tau + F\big(\wp(\backepsilon\eth, \backepsilon\beth)\big) \leq F\big(\wp(\eth, \beth)\big)$$

for $\eth, \beth \in \mho$.

Example 1.6. *Here are some examples of the functions $F : (0, \infty) \to \mathbb{R}$.*

(1) $F(\eth) = \ln(\eth)$,
(2) $F(\eth) = \eth + \ln(\eth)$,
(3) $F(\eth) = -\frac{1}{\sqrt{\eth}}$
for $\eth > 0$.

Lemma 1.7. *[18] Let $(\mho, \wp)$ be a MS and $\mu_1, \mu_2 \in CL(\mho)$ with $H(\mu_1, \mu_2) > 0$. Then, for each $h > 1$ and for each $\eth \in \mu_1$, $\exists\ \beth = \beth(\eth) \in \mu_2$ such that $\wp(\eth, \beth) < hH(\mu_1, \mu_2)$.*

2. Main Results

In this section, we obtain common fixed point results for fuzzy mappings from $\mho$ into $F(\mho)$ satisfying generalized F-contraction conditions in the setting of complete metric space.

Theorem 2.1. *Let $(\mho, \wp)$ be a complete metric space and let $\Game_1, \Game_2 : \mho \to F(\mho)$ and for each $\eth, \beth \in \mho$, $\exists\ \alpha_{\Game_1}(\eth), \alpha_{\Game_2}(\beth) \in (0,1]$ such that $[\Game_1\eth]_{\alpha_{\Game_1}(\eth)}$, $[\Game_2\beth]_{\alpha_{\Game_2}(\beth)} \in CB(\mho)$. Assume that there exists a continuous from the right function $F \in \digamma$ and $\tau > 0$ such that*

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})) \leq F\left(\sigma\left(\begin{array}{c} \wp(\eth,\beth), \wp(\eth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}), \wp(\beth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \\ \wp(\eth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \wp(\beth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}) \end{array}\right)\right) \tag{2.1}$$

for all $\eth, \beth \in \mho$ with $H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}) > 0$. Then there exists $\eth^* \in \mho$ such that

$$\eth^* \in [\Game_1\eth^*]_{\alpha_{\Game_1}(\eth^*)} \cap [\Game_2\eth^*]_{\alpha_{\Game_2}(\eth^*)}.$$

Remark 2.2. *From (2.1), we have*

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})) \leq F\left(\sigma\left(\begin{array}{c} \wp(\eth,\beth), \wp(\eth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}), \wp(\beth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \\ \wp(\eth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \wp(\beth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}) \end{array}\right)\right)$$

which implies that

$$\begin{aligned} F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})) &\leq F\left(\sigma\left(\begin{array}{c} \wp(\eth,\beth), \wp(\eth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}), \wp(\beth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \\ \wp(\eth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \wp(\beth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}) \end{array}\right)\right) - 2\tau \\ &< F\left(\sigma\left(\begin{array}{c} \wp(\eth,\beth), \wp(\eth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}), \wp(\beth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \\ \wp(\eth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \wp(\beth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}) \end{array}\right)\right) \end{aligned}$$

Since F is non-decreasing, we obtain

$$H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}) < \sigma\left(\begin{array}{c} \wp(\eth,\beth), \wp(\eth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}), \wp(\beth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \\ \wp(\eth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \wp(\beth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}) \end{array}\right)$$

for all $\eth, \beth \in \mho$ with $H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})$.

Proof. Let $\eth_0 \in \mho$, then by hypothesis, $\exists\ \alpha_{\Game_1}(\eth_0) \in (0,1]$ such that $[\Game_1\eth_0]_{\alpha_{\Game_1}(\eth_0)} \neq \emptyset$ and $[\Game_1\eth_0]_{\alpha_{\Game_1}(\eth_0)} \in CB(\mho)$. Let $\eth_1 \in [\Game_1\eth_0]_{\alpha_{\Game_1}(\eth_0)}$. For this $\eth_1$ there exists $\alpha_{\Game_2}(\eth_1) \in (0,1]$ such that $[\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)} \neq \emptyset$ and $[\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)} \in CB(\mho)$ such that

$$\begin{aligned} 2\tau + F\left(\wp\left(\eth_1, [\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}\right)\right) &\leq 2\tau + F\left(H\left([\Game_1\eth_0]_{\alpha_{\Game_1}(\eth_0)}, [\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}\right)\right) \\ &\leq F\left(\sigma\left(\begin{array}{c} \wp(\eth_0,\eth_1), \wp(\eth_0, [\Game_1\eth_0]_{\alpha_{\Game_1}(\eth_0)}), \wp(\eth_1, [\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}), \\ \wp(\eth_0, [\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}), \wp(\eth_1, [\Game_1\eth_0]_{\alpha_{\Game_1}(\eth_0)}) \end{array}\right)\right) \\ &\leq F\left(\sigma\left(\begin{array}{c} \wp(\eth_0,\eth_1), \wp(\eth_0,\eth_1), \wp(\eth_1, [\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}), \\ \wp(\eth_0, [\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}), 0 \end{array}\right)\right) \end{aligned}$$

and so

$$\wp\left(\eth_1, [\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}\right) < \sigma\left(\begin{array}{c} \wp(\eth_0,\eth_1), \wp(\eth_0,\eth_1), \wp(\eth_1, [\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}), \\ \wp(\eth_0, [\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}), 0 \end{array}\right)$$

Then Lemma 1.3 gives that $\wp\left(\eth_1, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right) < \wp(\eth_0, \eth_1)$. Thus, we obtain

$$\begin{aligned}
2\tau + F\left(\wp\left(\eth_1, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right)\right) &\leq 2\tau + F\left(H\left([\backepsilon_1\eth_0]_{\alpha_{\backepsilon_1}(\eth_0)}, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right)\right) \\
&\leq F\left(\sigma\left(\begin{array}{c} \wp(\eth_0, \eth_1), \wp(\eth_0, [\backepsilon_1\eth_0]_{\alpha_{\backepsilon_1}(\eth_0)}), \wp(\eth_1, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}), \\ \wp(\eth_0, \eth_1) + \wp(\eth_1, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}), 0 \end{array}\right)\right) \\
&< F\left(\sigma\left(\begin{array}{c} \wp(\eth_0, \eth_1), \wp(\eth_0, \eth_1), \wp(\eth_0, \eth_1), \\ 2\wp(\eth_0, \eth_1), 0 \end{array}\right)\right) \\
&\leq F\left(\wp(\eth_0, \eth_1)\sigma(1, 1, 1, 2, 0)\right) \\
&\leq F\left(\wp(\eth_0, \eth_1)\right)
\end{aligned}$$

Thus we have

$$2\tau + F\left(\wp\left(\eth_1, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right)\right) \leq F\left(\wp(\eth_0, \eth_1)\right) \tag{2.2}$$

□

Since $F \in \mathcal{F}$ is right hand continuous function, so $\exists\, h > 1$ such that

$$F\left(hH\left([\backepsilon_1\eth_0]_{\alpha_{\backepsilon_1}(\eth_0)}, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right)\right) < F\left(H\left([\backepsilon_1\eth_0]_{\alpha_{\backepsilon_1}(\eth_0)}, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right)\right) + \tau. \tag{2.3}$$

Next as

$$\wp\left(\eth_1, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right) \leq H\left([\backepsilon_1\eth_0]_{\alpha_{\backepsilon_1}(\eth_0)}, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right) < hH\left([\backepsilon_1\eth_0]_{\alpha_{\backepsilon_1}(\eth_0)}, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right)$$

by Lemma 1.7, there exists $\eth_2 \in [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}$ (obviously, $\eth_2 \neq \eth_1$) such that

$$\wp(\eth_1, \eth_2) \leq hH\left([\backepsilon_1\eth_0]_{\alpha_{\backepsilon_1}(\eth_0)}, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right). \tag{2.4}$$

Thus by (2.3) and (2.4), we have

$$F\left(\wp(\eth_1, \eth_2)\right) \leq F\left(hH\left([\backepsilon_1\eth_0]_{\alpha_{\backepsilon_1}(\eth_0)}, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right)\right) < F\left(H\left([\backepsilon_1\eth_0]_{\alpha_{\backepsilon_1}(\eth_0)}, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right)\right) + \tau$$

which implies by (2.2) that

$$\begin{aligned}
2\tau + F\left(\wp(\eth_1, \eth_2)\right) &\leq 2\tau + F\left(H\left([\backepsilon_1\eth_0]_{\alpha_{\backepsilon_1}(\eth_0)}, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right)\right) + \tau \\
&\leq F\left(\wp(\eth_0, \eth_1)\right) + \tau
\end{aligned}$$

Thus we have

$$\tau + F\left(\wp(\eth_1, \eth_2)\right) \leq F(\wp(\eth_0, \eth_1)). \tag{2.5}$$

For this $\eth_2$, there exists $\alpha_{\backepsilon_1}(\eth_2) \in (0, 1]$ such that $[\backepsilon_1\eth_2]_{\alpha_{\backepsilon_1}(\eth_2)} \neq \emptyset$ and $[\backepsilon_1\eth_2]_{\alpha_{\backepsilon_1}(\eth_2)} \in CB(\mho)$. Thus we have

$$\begin{aligned}
2\tau + F\left(\wp\left(\eth_2, [\backepsilon_1\eth_2]_{\alpha_{\backepsilon_1}(\eth_2)}\right)\right) &\leq 2\tau + F\left(H\left([\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}, [\backepsilon_1\eth_2]_{\alpha_{\backepsilon_1}(\eth_2)}\right)\right) \\
&= 2\tau + F\left(H\left([\backepsilon_1\eth_2]_{\alpha_{\backepsilon_1}(\eth_2)}, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}\right)\right) \\
&\leq F\left(\sigma\left(\begin{array}{c} \wp(\eth_2, \eth_1), \wp(\eth_2, [\backepsilon_1\eth_2]_{\alpha_{\backepsilon_1}(\eth_2)}), \wp(\eth_1, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}), \\ \wp(\eth_2, [\backepsilon_2\eth_1]_{\alpha_{\backepsilon_2}(\eth_1)}), \wp(\eth_1, [\backepsilon_1\eth_2]_{\alpha_{\backepsilon_1}(\eth_2)}) \end{array}\right)\right) \\
&\leq F\left(\sigma\left(\begin{array}{c} \wp(\eth_2, \eth_1), \wp(\eth_2, [\backepsilon_1\eth_2]_{\alpha_{\backepsilon_1}(\eth_2)}, \wp(\eth_1, \eth_2), \\ 0, \wp(\eth_1, [\backepsilon_1\eth_2]_{\alpha_{\backepsilon_1}(\eth_2)} \end{array}\right)\right) \\
&= F\left(\sigma\left(\begin{array}{c} \wp(\eth_1, \eth_2), \wp(\eth_2, [\backepsilon_1\eth_2]_{\alpha_{\backepsilon_1}(\eth_2)}, \wp(\eth_1, \eth_2), \\ 0, \wp(\eth_1, [\backepsilon_1\eth_2]_{\alpha_{\backepsilon_1}(\eth_2)} \end{array}\right)\right)
\end{aligned}$$

and so

$$\wp\left(\eth_2, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right) < \sigma\left(\begin{array}{c}\wp(\eth_1,\eth_2), \wp(\eth_2, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}, \wp(\eth_1,\eth_2), \\ 0, \wp(\eth_1, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\end{array}\right)$$

Then Lemma 1.3 gives that $\wp\left(\eth_2, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right) < \wp(\eth_1,\eth_2)$. Thus, we obtain

$$\begin{array}{rcl} 2\tau + F\left(\wp\left(\eth_2, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right)\right) & \leq & 2\tau + F\left(H\left([\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right)\right) \\ & = & 2\tau + F\left(H\left([\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}, [\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}\right)\right) \\ & \leq & F\left(\sigma\left(\begin{array}{c}\wp(\eth_1,\eth_2), \wp(\eth_2, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}, \wp(\eth_1,\eth_2), \\ 0, \wp(\eth_1, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\end{array}\right)\right) \\ & \leq & F\left(\sigma\left(\begin{array}{c}\wp(\eth_1,\eth_2), \wp(\eth_1,\eth_2), \wp(\eth_1,\eth_2), \\ 0, 2\wp(\eth_1,\eth_2)\end{array}\right)\right) \\ & \leq & F\left(\wp(\eth_1,\eth_2)\sigma\left(1,1,1,0,2\right)\right) \\ & \leq & F\left(\wp(\eth_1,\eth_2)\right) \end{array}$$

Thus we have

$$2\tau + F\left(\wp\left(\eth_2, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right)\right) \leq F\left(\wp(\eth_1,\eth_2)\right) \tag{2.6}$$

Since $F \in \digamma$ is right hand continuous function, so $\exists\ h > 1$ such that

$$F\left(hH\left([\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right)\right) < F\left(H\left([\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right)\right) + \tau. \tag{2.7}$$

Next as

$$\wp\left(\eth_2, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right) \leq H\left([\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right) < hH\left([\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right)$$

by Lemma 1.7, there exists $\eth_3 \in [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}$ (obviously, $\eth_3 \neq \eth_2$) such that

$$\wp(\eth_2,\eth_3) \leq hH\left([\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right). \tag{2.8}$$

Thus by (2.7) and (2.8), we have

$$F\left(\wp(\eth_2,\eth_3)\right) \leq F\left(hH\left([\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right)\right) < F\left(H\left([\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right)\right) + \tau$$

which implies by (2.6) that

$$\begin{array}{rcl} 2\tau + F\left(\wp(\eth_2,\eth_3)\right) & \leq & 2\tau + F\left(H\left([\Game_2\eth_1]_{\alpha_{\Game_2}(\eth_1)}, [\Game_1\eth_2]_{\alpha_{\Game_1}(\eth_2)}\right)\right) + \tau \\ & \leq & F\left(\wp(\eth_1,\eth_2)\right) + \tau \end{array}$$

Thus we have

$$\tau + F\left(\wp(\eth_2,\eth_3)\right) \leq F(\wp(\eth_1,\eth_2)). \tag{2.9}$$

So, doing in this way, we get $\{\eth_j\}$ in $\mho$ such that $\eth_{2j+1} \in [\Game_1\eth_{2j}]_{\alpha_{\Game_1}(\eth_{2j})}$ and $\eth_{2j+2} \in [\Game_2\eth_{2j+1}]_{\alpha_{\Game_2}(\eth_{2j+1})}$ and

$$\tau + F(\wp(\eth_{2j+1},\eth_{2j+2})) \leq F(\wp(\eth_{2j},\eth_{2j+1})) \tag{2.10}$$

and

$$\tau + F(\wp(\eth_{2j+2},\eth_{2j+3})) \leq F(\wp(\eth_{2j+1},\eth_{2j+2})) \tag{2.11}$$

$\forall\ j \in \mathbb{N}$. By (2.9) and (2.10), we have

$$\tau + F(\wp(\eth_j,\eth_{j+1})) \leq F(\wp(\eth_{j-1},\eth_j)) \tag{2.12}$$

for all $j \in \mathbb{N}$. Therefore by (2.12), we have

$$\begin{aligned} F\left(\wp(\eth_j, \eth_{j+1})\right) &\leq F\left(\wp(\eth_{j-1}, \eth_j)\right) - \tau \leq F\left(\wp(\eth_{j-2}, \eth_{j-1})\right) - 2\tau \\ &\leq \ldots \leq F\left(\wp(\eth_0, \eth_1)\right) - j\tau. \end{aligned} \tag{2.13}$$

Letting $j \to \infty$, we obtain $\lim_{j\to\infty} F\left(\wp(\eth_j, \eth_{j+1})\right) = -\infty$ along with (F_2) gives

$$\lim_{j\to\infty} \wp(\eth_j, \eth_{j+1}) = 0.$$

Now by (F_3), there exists $r \in (0,1)$ such that

$$\lim_{j\to\infty} [\wp(\eth_j, \eth_{j+1})]^r F\left(\wp(\eth_j, \eth_{j+1})\right) = 0. \tag{2.14}$$

From (2.14) we have

$$\begin{aligned} & [\wp(\eth_j, \eth_{j+1})]^r F\left(\wp(\eth_j, \eth_{j+1})\right) - [\wp(\eth_j, \eth_{j+1})]^r F\left(\wp(\eth_0, \eth_1)\right) \\ \leq\ & [\wp(\eth_j, \eth_{j+1})]^r [F\left(\wp(\eth_0, \eth_1)\right) - j\tau] - [\wp(\eth_j, \eth_{j+1})]^r F\left(\wp(\eth_0, \eth_1)\right) \\ \leq\ & -j\tau[\wp(\eth_j, \eth_{j+1})]^r \leq 0. \end{aligned}$$

Letting $j \to \infty$ we get

$$\lim_{j\to\infty} j[\wp(\eth_j, \eth_{j+1})]^r = 0. \tag{2.15}$$

Hence $\lim_{j\to\infty} j^{\frac{1}{r}} \wp(\eth_j, \eth_{j+1}) = 0$ and $\exists\ j_1 \in \mathbb{N}$ such that $j^{\frac{1}{r}} \wp(\eth_j, \eth_{j+1}) \leq 1$, $\forall\ j \geq j_1$. So we get

$$\wp(\eth_j, \eth_{j+1}) \leq j^{\frac{1}{r}}$$

for all $j \geq j_1$. Now taking $m, j \in \mathbb{N}$ such that $m > j \geq j_1$, we have

$$\begin{aligned} \wp\left(\eth_j, \eth_m\right) &\leq \wp\left(\eth_j, \eth_{j+1}\right) + \wp\left(\eth_{j+1}, \eth_{j+2}\right) + \ldots + \wp\left(\eth_{m-1}, \eth_m\right) \\ &\leq \sum_{i=j}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

Since $\sum_{i=j}^{\infty} \frac{1}{i^{1/r}}$ is convergent, so we get $\wp\left(\eth_j, \eth_m\right) \to 0$ as $j, m \to \infty$. Therefore $\{\eth_j\}$ is a Cauchy sequence in $\mho$. Since $\mho$ is complete, there exists an element $\eth^* \in \mho$ such that $\eth_j \to \eth^*$ as $j \to \infty$ that is

$$\lim_{j\to\infty} \eth_j = \eth^*. \tag{2.16}$$

Now, we prove that $\eth^* \in [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}$. We suppose on the contrary that $\eth^* \notin [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}$, then there exist a $j_0 \in \mathbb{N}$ and a subsequence $\{\eth_{j_k}\}$ of $\{\eth_j\}$ such that $\wp(\eth_{2j_k+1}, [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}) > 0$ for all $j_k \geq j_0$. Now, using (2.1) with $\eth = \eth_{2j_k+1}$ and $\beth = \eth^*$. Taking Remark 2.2 into account, we have

$$\begin{aligned} \wp(\eth_{2j_k+1}, [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}) &\leq H([\eth_1\eth_{2j_k}]_{\alpha_{\eth_1}(\eth_{2j_k})}, [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}) \\ &\leq \sigma\begin{pmatrix} \wp(\eth_{2j_k}, \eth^*), \wp(\eth_{2j_k}, [\eth_1\eth_{2j_k}]_{\alpha_{\eth_1}(\eth_{2j_k})}), \wp(\eth^*, [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}), \\ \wp(\eth_{2j_k}, [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}), \wp(\eth^*, [\eth_1\eth_{2j_k}]_{\alpha_{\eth_1}(\eth_{2j_k})}) \end{pmatrix} \\ &\leq \sigma\begin{pmatrix} \wp(\eth_{2j_k}, \eth^*), \wp(\eth_{2j_k}, \eth_{2j_k+1}), \wp(\eth^*, [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}), \\ \wp(\eth_{2j_k}, [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}), \wp(\eth^*, \eth_{2j_k+1}) \end{pmatrix}. \end{aligned}$$

Passing to limit as $j \to \infty$ in the above inequality, we obtain

$$\wp(\eth^*, [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}) \leq \sigma\left(0, 0, \wp(\eth^*, [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}), \wp(\eth^*, [\eth_2\eth^*]_{\alpha_{\eth_2}(\eth^*)}), 0\right)$$

which implies by Lemma 1.3 that

$$0 < \wp(\eth^*, [\Game_2\eth^*]_{\alpha_{\Game_2}(\eth^*)}) < 0$$

which is a contradiction. Hence $\wp(\eth^*, [\Game_2\eth^*]_{\alpha_{\Game_2}(\eth^*)}) = 0$. Since $[\Game_2\eth^*]_{\alpha_{\Game_2}(\eth^*)}$ is closed, we deduce that $\eth^* \in [\Game_2\eth^*]_{\alpha_{\Game_2}(\eth^*)}$. Similarly, one can easily prove that $\eth^* \in [\Game_1\eth^*]_{\alpha_{\Game_1}(\eth^*)}$. Thus $\eth^* \in [\Game_1\eth^*]_{\alpha_{\Game_1}(\eth^*)} \cap [\Game_2\eth^*]_{\alpha_{\Game_2}(\eth^*)}$.

Note: From now onwrds, we consider $(\mho, \wp)$ as a complete metric space and $F \in \digamma$ as a right hand continuous function.

Example 2.3. *Let $\mho = [0,1]$ and define $\wp : \mho \times \mho \to \mathbb{R}^+$ as follows:*

$$\wp(\eth, \sqsupset) = |\eth - \sqsupset| .$$

Then $(\mho, \wp)$ is a complete metric space. Define $\Game_1, \Game_2 : \mho \to F(\mho)$, as follows:

$$\Game_1(\eth)(t) = \begin{cases} \alpha \text{ if } 0 \le t \le \frac{\eth}{16} \\ \frac{\alpha}{2} \text{ if } \frac{\eth}{16} < t \le \frac{\eth}{9} \\ \frac{\alpha}{3} \text{ if } \frac{\eth}{9} < t \le \frac{\eth}{4} \\ \frac{\alpha}{5} \text{ if } \frac{\eth}{4} < t \le 1 \end{cases},$$

and

$$\Game_2(\eth)(t) = \begin{cases} \alpha \text{ if } 0 \le t \le \frac{\eth}{8} \\ \frac{\alpha}{3} \text{ if } \frac{\eth}{8} < t \le \frac{\eth}{5} \\ \frac{\alpha}{4} \text{ if } \frac{\eth}{5} < t \le \frac{\eth}{2} \\ \frac{\alpha}{7} \text{ if } \frac{\eth}{2} < t \le 1. \end{cases},$$

for $\alpha \in [0,1]$ and $\eth \in \mho$ such that

$$\begin{aligned} [\Game_2\eth]_\alpha &= \left[0, \frac{\eth}{8}\right], \\ [\Game_1\eth]_\alpha &= \left[0, \frac{\eth}{16}\right]. \end{aligned}$$

Let $\sigma : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ be defined by $\sigma(\eth_1, \eth_2, \eth_3, \eth_4, \eth_5) = \max\{\eth_1, \eth_2, \eth_3, \eth_4, \eth_5\}$ and $F(t) = \ln(t)$ for $t > 0$. Then $\exists\ \tau = \ln(3)$. All the hypothesis of our main Theorem 2.1 to obtain $0 \in [\Game_1 0]_\alpha \cap [\Game_2 0]_\alpha$.

Corollary 2.4. *Let $\Game : (\mho, \wp) \to F(\mho)$ and for each $\eth, \sqsupset \in \mho$, there exist $\alpha_\Game(\eth), \alpha_\Game(\sqsupset) \in (0,1]$ such that $[\Game\eth]_{\alpha_\Game(\eth)}$, $[\Game\sqsupset]_{\alpha_\Game(\sqsupset)} \in CB(\mho)$. Assume that there exists a $F \in \digamma$ and $\tau > 0$ such that*

$$2\tau + F(H([\Game\eth]_{\alpha_\Game(\eth)}, [\Game\sqsupset]_{\alpha_\Game(\sqsupset)})) \le F\left(\sigma\left(\begin{array}{c} \wp(\eth, \sqsupset), \wp(\eth, [\Game\eth]_{\alpha_\Game(\eth)}), \wp(\sqsupset, [\Game\sqsupset]_{\alpha_\Game(\sqsupset)}), \\ \wp(\eth, [\Game\sqsupset]_{\alpha_\Game(\sqsupset)}), \wp(\sqsupset, [\Game\eth]_{\alpha_\Game(\eth)}) \end{array}\right)\right)$$

for all $\eth, \sqsupset \in \mho$ with $H([\Game\eth]_{\alpha_\Game(\eth)}, [\Game\sqsupset]_{\alpha_\Game(\sqsupset)}) > 0$. Then there exists $\eth^* \in \mho$ such that $\eth^* \in [\Game\eth^*]_{\alpha_\Game(\eth^*)}$.

Corollary 2.5. *Let $\Game$: $(\mho, \wp) \to F(\mho)$ and for each $\eth, \sqsupset \in \mho$, there exist $\alpha_\Game(\eth), \alpha_\Game(\sqsupset) \in (0,1]$ such that $[\Game\eth]_{\alpha_\Game(\eth)}$, $[\Game\sqsupset]_{\alpha_\Game(\sqsupset)} \in CB(\mho)$. Assume that there exists a $F \in \digamma$ and $\tau > 0$ such that*

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\sqsupset]_{\alpha_{\Game_2}(\sqsupset)})) \le F(\wp(\eth, \sqsupset)) .$$

for all $\eth, \sqsupset \in \mho$ with $H([\Game\eth]_{\alpha_\Game(\eth)}, [\Game\sqsupset]_{\alpha_\Game(\sqsupset)}) > 0$. Then there exists $\eth^* \in \mho$ such that $\eth^* \in [\Game\eth^*]_{\alpha_\Game(\eth^*)}$.

3. Consequences for Fuzzy Fixed Points

Remark 3.1. *By Example 1.2 (i), we get the following contractive condition*

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\sqsupset]_{\alpha_{\Game_2}(\sqsupset)})) \le F(\wp(\eth, \sqsupset)) .$$

By Example 1.2 (ii), we get the following contractive condition

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})) \leq F\left(\wp(\eth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}) + \wp(\beth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})\right).$$

By Example 1.2 (iii), we get the following contractive condition

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})) \leq F\left(\wp(\eth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}) + \wp(\beth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)})\right).$$

By Example 1.2 (iv), we get the following contractive condition

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})) \leq F\left(\daleth_1\wp(\eth, \beth) + \daleth_2\wp(\eth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}) + \daleth_3\wp(\beth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})\right)$$

By Example 1.2 (v), we get the following contractive condition

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})) \leq F\left(\begin{array}{c} \daleth_1\wp(\eth, \beth) + \daleth_2\wp(\eth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}) + \daleth_3\wp(\beth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}) \\ +\daleth_4\wp(\eth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}) + \daleth_5\wp(\beth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}) \end{array}\right)$$

By Example 1.2 (vi), we get the following contractive condition

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})) \leq F\left(\daleth_1\wp(\eth, \beth) + \daleth_5\wp(\beth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)})\right)$$

By Example 1.2 (vii), we get the following contractive condition

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})) \leq F\left(\max\left\{\begin{array}{c} \wp(\eth, \beth), \wp(\eth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}), \wp(\beth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \\ \frac{\wp(\eth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}) + \wp(\beth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)})}{2} \end{array}\right\}\right)$$

By Example 1.2 (viii), we get the following contractive condition

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})) \leq F\left(\max\left\{\begin{array}{c} \wp(\eth, \beth), \wp(\eth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}), \wp(\beth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \\ \wp(\eth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}), \wp(\beth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}) \end{array}\right\}\right)$$

By Example 1.2 (ix), we get the following contractive condition

$$2\tau + F(H([\Game_1\eth]_{\alpha_{\Game_1}(\eth)}, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})) \leq F\left(\max\left\{\begin{array}{c} \wp(\eth, \beth), \frac{\wp(\eth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)}) + \wp(\beth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)})}{2}, \\ \frac{\wp(\eth, [\Game_2\beth]_{\alpha_{\Game_2}(\beth)}) + \wp(\beth, [\Game_1\eth]_{\alpha_{\Game_1}(\eth)})}{2} \end{array}\right\}\right)$$

For all above results, there exists $\eth^* \in \mho$ such that $\eth^* \in [\Game_1\eth^*]_{\alpha_{\Game_1}(\eth^*)} \cap [\Game_2\eth^*]_{\alpha_{\Game_2}(\eth^*)}$ by Theorem 2.1.

4. Consequences for Multivalued Mappings

Theorem 4.1. *Let $\mathcal{P}, (\mho, \wp) : \mho \to CB(\mho)$. If $\exists\ \tau > 0$ and $F \in \digamma$ such that*

$$2\tau + F(H(\mathcal{P}\eth, \mathcal{Q}\beth)) \leq F\left(\sigma\left(\begin{array}{c} \wp(\eth, \beth), \wp(\eth, \mathcal{P}\eth), \wp(\beth, \mathcal{Q}\beth), \\ \wp(\eth, \mathcal{Q}\beth), \wp(\beth, \mathcal{P}\eth) \end{array}\right)\right)$$

for all $\eth, \beth \in \mho$ with $H(\mathcal{P}\eth, \mathcal{Q}\beth) > 0$. Then $\exists\ \eth^* \in \mho$ such that $\eth^* \in \mathcal{P}\eth^* \cap \mathcal{Q}\eth^*$.

Proof. Consider $\alpha : \mho \to (0, 1]$ and $\Game_1, \Game_2 : \mho \to F(\mho)$ defined by

$$\Game_1(\eth)(t) = \begin{cases} \alpha(\eth), & \text{if } t \in \mathcal{P}\eth, \\ 0, & \text{if } t \notin \mathcal{P}\eth \end{cases}$$

and

$$\Game_2(\eth)(t) = \begin{cases} \alpha(\eth), & \text{if } t \in \mathcal{Q}\eth, \\ 0, & \text{if } t \notin \mathcal{Q}\eth. \end{cases}$$

Then

$$[\partial_1 \eth]_{\alpha(\eth)} = \{t : \partial_1(\eth)(t) \geq \alpha(\eth)\} = \mathcal{P}\eth \quad \text{and} \qquad [\partial_2 \eth]_{\alpha(\eth)} = \{t : \partial_2(\eth)(t) \geq \alpha(\eth)\} = \mathcal{Q}\eth.$$

Thus, Theorem 2.1 can be applied to get $\eth^* \in \mho$ such that $\eth^* \in [\partial_1 \eth^*]_{\alpha_{\partial_1}(\eth^*)} \cap [\partial_2 \eth^*]_{\alpha_{\partial_2}(\eth^*)} = \mathcal{P}\eth^* \cap \mathcal{Q}\eth^*$. □

Corollary 4.2. *Let $\mathcal{Q} : (\mho, \wp) \to CB(\mho)$. If $\exists\ \tau > 0$ and $F \in \digamma$ such that*

$$2\tau + F(H(\mathcal{Q}\eth, \mathcal{Q}\beth)) \leq F\left(\sigma\left(\begin{array}{c} \wp(\eth, \beth), \wp(\eth, \mathcal{Q}\eth), \wp(\beth, \mathcal{Q}\beth), \\ \wp(\eth, \mathcal{Q}\beth), \wp(\beth, \mathcal{Q}\eth) \end{array}\right)\right)$$

for all $\eth, \beth \in \mho$ with $H(\mathcal{Q}\eth, \mathcal{Q}\beth) > 0$. Then $\exists\ \eth^* \in \mho$ such that $\eth^* \in \mathcal{Q}\eth^*$.

Corollary 4.3. *Let $\mathcal{Q} : (\mho, \wp) \to CB(\mho)$. If $\exists\ \tau > 0$ and $F \in \digamma$ such that*

$$2\tau + F(H(\mathcal{P}\eth, \mathcal{Q}\beth)) \leq F\left(\wp(\eth, \beth)\right).$$

for all $\eth, \beth \in \mho$ with $H(\mathcal{Q}\eth, \mathcal{Q}\beth) > 0$. Then $\exists\ \eth^* \in \mho$ such that $\eth^* \in \mathcal{Q}\eth^*$.

Following the same procedure and Example 1.2, we have the following the remarks.

Remark 4.4. *By Example 1.2 (ii), we get the following contractive condition*

$$2\tau + F(H(\mathcal{P}\eth, \mathcal{Q}\beth)) \leq F\left(\wp(\eth, \mathcal{P}\eth) + \wp(\beth, \mathcal{Q}\beth)\right).$$

By Example 1.2 (iii), we get the following contractive condition

$$2\tau + F(H(\mathcal{P}\eth, \mathcal{Q}\beth)) \leq F\left(\wp(\eth, \mathcal{Q}\beth) + \wp(\beth, \mathcal{P}\eth)\right).$$

By Example 1.2 (iv), we get the following contractive condition

$$2\tau + F(H(\mathcal{P}\eth, \mathcal{Q}\beth)) \leq F\left(\alpha\wp(\eth, \beth) + \beta\wp(\eth, \mathcal{P}\eth) + \gamma\wp(\beth, \mathcal{Q}\beth)\right)$$

By Example 1.2 (v), we get the following contractive condition

$$2\tau + F(H(\mathcal{P}\eth, \mathcal{Q}\beth)) \leq F\left(\begin{array}{c} \alpha\wp(\eth, \beth) + \beta\wp(\eth, \mathcal{P}\eth) + \gamma\wp(\beth, \mathcal{Q}\beth) \\ +\delta\wp(\eth, \mathcal{Q}\beth) + L\wp(\beth, \mathcal{P}\eth) \end{array}\right).$$

By Example 1.2 (vi), we get the following contractive condition

$$2\tau + F(H(\mathcal{P}\eth, \mathcal{Q}\beth)) \leq F\left(\alpha\wp(\eth, \beth) + L\wp(\beth, \mathcal{P}\eth)\right)$$

By Example 1.2 (vii), we get the following contractive condition

$$2\tau + F(H(\mathcal{P}\eth, \mathcal{Q}\beth)) \leq F\left(\max\left\{\begin{array}{c} \wp(\eth, \beth), \wp(\eth, \mathcal{P}\eth), \wp(\beth, \mathcal{Q}\beth), \\ \frac{\wp(\eth, \mathcal{Q}\beth) + \wp(\beth, \mathcal{P}\eth)}{2} \end{array}\right\}\right).$$

By Example 1.2 (viii), we get the following contractive condition

$$2\tau + F(H(\mathcal{P}\eth, \mathcal{Q}\beth)) \leq F\left(\max\left\{\begin{array}{c} \wp(\eth, \beth), \wp(\eth, \mathcal{P}\eth), \wp(\beth, \mathcal{Q}\beth), \\ \wp(\eth, \mathcal{Q}\beth), \wp(\beth, \mathcal{P}\eth) \end{array}\right\}\right)$$

By Example 1.2 (ix), we get the following contractive condition

$$2\tau + F(H(\mathcal{P}\eth, \mathcal{Q}\beth)) \leq F\left(\max\left\{\wp(\eth, \beth), \frac{\wp(\eth, \mathcal{P}\eth) + \wp(\beth, \mathcal{Q}\beth)}{2}, \frac{\wp(\eth, \mathcal{Q}\beth) + \wp(\beth, \mathcal{P}\eth)}{2}\right\}\right).$$

For all above cases, $\mathcal{P}$ and $\mathcal{Q}$ have common fixed point by Theorem 4.1.

5. Applications to domain of words

Suppose $\Omega \neq \varnothing$ and be set of alphabets and Ω^{∞} be the collection of all finite sequences and infinite sequences ("words") over Ω, where we assume the understanding that $\emptyset$ is a member of Ω^{∞}. Furthermore, on Ω^{∞}, we take $\precsim$ as the prefix order which is given in this way

$$\zeta \precsim \gamma \iff \zeta \text{ is a prefix of } \gamma.$$

For each $\zeta \neq \emptyset$ and $\zeta \in \Omega^{\infty}$, $l(\zeta)$ presents the length of ζ. Now for each $\zeta \neq \emptyset$ and $l(\emptyset) = 0$, $l(\zeta) \in [0, \infty]$ and $\zeta, \gamma \in \Omega^{\infty}$, $\zeta \sqcap \gamma$ denotes the common prefix of ζ and γ. Evidently, $\zeta = \gamma \iff \zeta \precsim \gamma$ and $\gamma \precsim \zeta$ and $l(\zeta) = l(\gamma)$. Then, the the Baire metric $\sigma_{\precsim}$ is given on $\Omega^{\infty} \times \Omega^{\infty}$ by

$$\begin{cases} \sigma_{\precsim}(\zeta, \gamma) = 0, \text{ if } \zeta = \gamma \\ \sigma_{\precsim}(\zeta, \gamma) = 2^{-l(\zeta \sqcap \gamma)}, \text{ otherwise} \end{cases}$$

such that the metric space $(\Omega^{\infty}, \sigma_{\precsim})$ is complete.

Exactly, we deal with the following recurrence relation:

$$\mathfrak{R}(1) = 0 \quad \text{and} \quad \mathfrak{R}(j) = \frac{2(j-1)}{j} + \frac{j+1}{j}\mathfrak{R}(j-1), \quad j \geq 2. \tag{5.1}$$

Consider as an alphabet $\Omega = \mathbb{R}^{+}$. We accomplice to $\mathfrak{R}$ the functional $\Phi : \Omega^{\infty} \to \Omega^{\infty}$ defined by

$$(\Phi(\zeta))_1 = \mathfrak{R}(1)$$

and

$$(\Phi(\zeta))_j = \frac{2(j-1)}{j} + \frac{j+1}{j}\zeta_{j-1}$$

$\forall\, j \geq 2$ (if $\zeta \in \Omega^{\infty}$ has length $j < \infty$, we write $\zeta := \zeta_1\zeta_2...\zeta_{j,}$ and if ζ is a word, we write $\zeta := (\zeta_1\zeta_2....)$. It follows by the formation that $l(\Phi(\zeta)) = l(\zeta) + 1$, $\forall\, \zeta \in \Omega^{\infty}$ and $l(\Phi(\zeta)) = +\infty$ whenever $l(\zeta) = +\infty$. We will prove that the functional Φ has an fuzzy fixed point by an application of Corollary 2.5. Let $\mathcal{P} : \Omega^{\infty} \to \Im(\Omega^{\infty})$ be the fuzzy mapping given by

$$\mathcal{P}_{\zeta} = (\Phi(\zeta))_{\alpha} \text{ for all } \zeta \in \Omega^{\infty} \text{ and } \alpha \in (0, 1].$$

and analyze these cases:

Case 01: If $\zeta = \gamma$, then we have

$$\mathcal{H}_{\precsim}((\Phi(\zeta))_{\alpha}, (\Phi(\zeta))_{\alpha}) = 0 = \sigma_{\precsim}(\zeta, \zeta).$$

Case 02: If $\zeta \neq \gamma$, then we write

$$\begin{aligned} \mathcal{H}_{\precsim}((\Phi(\zeta))_{\alpha}, (\Phi(\gamma))_{\alpha}) &= \sigma_{\precsim}((\Phi(\zeta))_{\alpha}, (\Phi(\gamma))_{\alpha}) = 2^{-(l((\Phi(\zeta))_{\alpha} \sqcap (\Phi(\gamma))_{\alpha})} \\ &\leq 2^{-(l(\Phi(\zeta \sqcap \gamma))_{\alpha})} = 2^{-(l(\zeta \sqcap \gamma)+1)} \\ &= \frac{1}{2} 2^{-l(\zeta \sqcap \gamma)} = \frac{1}{2}\sigma_{\precsim}(\zeta, \gamma). \end{aligned}$$

Instantly, we can achieve that all the assertions of the Corollary 2.5 are satisfied with $F(t) = \ln t$ and $\tau = -\ln \frac{1}{\sqrt{2}} > 0$. Hence, the fuzzy mapping $\mathcal{P}$ has a fuzzy fixed point $\zeta = \zeta_1\zeta_2... \in \Omega^{\infty}$ that is, $\zeta \in (\mathcal{P}_{\zeta})_{\alpha_L}$. Also, in the light of the definition of $\mathcal{P}, \zeta$ is a fixed point of Φ, and thus, ζ is the solution of (5.1). We have

$$\zeta_1 = 0,$$

$$\zeta_j = \frac{2(j-1)}{j} + \frac{j+1}{j}\zeta_{j-1}, \quad j \geq 2.$$

References

1. H. Adibi, Y.J. Cho, D. O'Regan, R. Saadati, Common fixed point theorems in L-fuzzy metric spaces, Appl. Math. Comput. 182 (2006) 820–828.
2. J. Ahmad, N. Hussain, A. R. Khan, A. Azam, Fixed Point Results for Generalized Multi-valued Contractions, The Journal of Nonlinear Sciences and Applications, (2015), 909-918.
3. J. Ahmad, A. Azam, S. Romaguera, On locally contractive fuzzy set-valued mappings, Journal of Inequalities and Applications 2014, 2014:74
4. A. Al-Rawashdeh and J. Ahmad, Common Fixed Point Theorems for JS- Contractions, Bulletin of Mathematical Analysis and Applications,Volume 8 Issue 4(2016), Pages 12-22.
5. SC. Arora and V. Sharma, Fixed points for fuzzy mappings, Fuzzy Sets Syst. 110(2000), 127-130
6. A. Azam, I. Beg, Common fixed points of fuzzy maps, Mathematical and Computer Modelling, 49 (2009), 1331-1336.
7. A. Azam, M. Arshad, P. Vetro, On a pair of fuzzy φ-contractive mappings. Math. Comput. Model. 52, 207-214 (2010)
8. A. Azam, Fuzzy Fixed Points of Fuzzy Mappings via a Rational Inequality, Hacettepe Journal of Mathematics and Statistics. Volume 40 (3), 421 – 431 (2011).
9. R.K. Bose, D. Sahani, Fuzzy mappings and fixed point theorems, Fuzzy Sets and Systems, 21 (1987), 53-58.
10. S.S. Chang, Y.J. Cho, B.S. Lee, J.S. Jung, S.M. Kang, Coincidence point and minimization theorems in fuzzy metric spaces, Fuzzy Sets Syst. 88 (1997) 119– 128.
11. YJ. Cho and N. Petrot, Existence theorems for fixed fuzzy points with closed α-cut sets in complete metric spaces, Commun. Korean Math. Soc. 26 (2011), No. 1, pp. 115-124.
12. A. Constantin, A random fixed point theorem for multifunctions, Stochastic Anal. Appl. 12(1), 65-73 (1994)
13. S. Heilpern, Fuzzy mappings and fixed point theorem, J. Math. Anal. Appl. 83 (1981), no. 2, 566-569.
14. N. Hussain, J. Ahmad, L. Ćirić and A. Azam, Coincidence point theorems for generalized contractions with application to integral equations, Fixed Point Theory and Applications, (2015) 2015:78
15. N. Hussain, V. Parvaneh, B. Samet and C. Vetro, Some fixed point theorems for generalized contractive mappings in complete metric spaces, Fixed Point Theory and Applications, 20152015:185
16. N. Hussain, J. Ahmad and A. Azam, On Suzuki-Wardowski type fixed point theorems, J. Nonlinear Sci. Appl. 8 (2015), 1095-1111.
17. N. Hussain, J. Ahmad, and A. Azam, Generalized fixed point theorems for multi-valued α-ψ contractive mappings, Journal of Inequalities and Applications, vol. 2014, article 348, 2014.
18. H. Isik, Fractional Differential Inclusions with a new class of set-valued contractions, https://arxiv.org/abs/1807.05427v1
19. M. Jleli, B. Samet, A new generalization of the Banach contraction principle. J. Inequal. Appl. 2014, 38 (2014)
20. J. R. Roshan, V. Parvaneh, Z. Kadelburg , Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces, J. Nonlinear Sci. Appl. 2014, 7, 229–245
21. Z. Shi-sheng, Fixed point theorems for fuzzy mappings (II), Appl. Math. Mech. 7(2), 147-152 (1986).
22. A. Sîntamarian , Integral inclusions of Fredholm type relative to multivalued φ-contractions, Seminar Fixed Point Theory Cluj Napoca 2002, 3, 361–368
23. D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory and Appl. 2012, 2012:94

Durdana Lateef,
Department of Mathematics,
College of Science, Taibah University,
Al Madina Al Munawwara,
Madina 41411, Saudi Arabia.
E-mail address: drdurdanamaths@gmail.com