Three Nontrivial Solutions of Boundary Value Problems for Semilinear $\Delta_\gamma$–Laplace Equation

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ABSTRACT: In this paper, we study the multiplicity of weak solutions to the boundary value problem

$$\Delta_\gamma u + f(x, u) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$

where $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^N$ ($N \geq 2$) and $\Delta_\gamma$ is the subelliptic operator of the type

$$\Delta_\gamma := \sum_{j=1}^N \partial_{x_j} \left( \gamma_j^2 \partial_{x_j} \right), \quad \partial_{x_j} := \frac{\partial}{\partial x_j}, \quad \gamma = (\gamma_1, \gamma_2, ..., \gamma_N),$$

the nonlinearity $f(x, \xi)$ is subcritical growth and may be not satisfy the Ambrosetti-Rabinowitz (AR) condition. We establish the existence of three nontrivial solutions by using Morse theory.

Key Words: Semilinear degenerate elliptic equations, Morse theory, Three solutions, Multiple solutions.

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1. Introduction

In the last decades, the boundary value problem for semilinear elliptic equations

$$-\Delta u = f(x, u), \quad x \in \Omega, \quad u \in H^1_0(\Omega),$$

has been studied by many authors, see, for example [1,20] and the references therein. The following (AR) condition introduced in [1]

(AR) For some $\theta > 2$ and $R > 0$, we have

$$\theta F(x, \xi) \leq f(x, \xi)\xi, \quad \forall |\xi| \geq R, \quad \forall x \in \Omega,$$

where $F(x, \xi) = \int_0^\xi f(x, \tau) d\tau$, plays an important role in their studies. Boundary value problems for nonlinear degenerate elliptic differential equations were treated in [10] and then subsequently in [8,5]. In [25,26], the critical exponent phenomenon was observed for a model of the Grushin type operators. The results were then generalized in [23] to a large class of semilinear degenerate elliptic differential equations. Recently, in [23,24] the second author of this paper and his collaborator have extended the research to a more complicated class of nonlinear degenerate elliptic differential operators. Very recently, the authors of [11] investigated the $\Delta_\gamma$–Laplace operator under the additional assumption that the operator is homogeneous of degree two with respect to a semigroup of dilations in $\mathbb{R}^N$. Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [27,28] (see also some recent results in [2,3,11,12,13,14,15,16,17,18,19,22,24,26]).

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In this paper, we study multiplicity of weak solutions to the following problem

\[
\Delta_\gamma u + f(x, u) = 0 \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial\Omega,
\]

where \(\Omega\) is a bounded domain with smooth boundary in \(\mathbb{R}^N\), \(\Delta_\gamma\) (see the definition of this function space below) and \(f(x, \xi) : \Omega \times \mathbb{R} \to \mathbb{R}\) such that \(f(x, 0) = 0\).

Let \(F(x, \xi) = \int_0^\xi f(x, \tau)d\tau\) and suppose that the non-linearity \(f\) satisfies the following conditions:

(A1) \(f \in C(\bar{\Omega} \times \mathbb{R})\) with \(f(x, 0) = 0\) and satisfies the improved subcritical polynomial growth condition, i.e.

\[\lim_{\xi \to \infty} \frac{f(x, \xi)}{|\xi|^{2\gamma-1}} = 0 \quad \text{uniformly for} \quad x \in \Omega,\]

where \(2\gamma := 2\bar{N}/(\bar{N} - 2)\);

(A2) \(\lim_{|\xi| \to 0} \frac{f(x, \xi)}{\xi} = p(x)\), uniformly for \(x \in \Omega\), where \(p \in L^\infty(\Omega)\) satisfies \(p(x) \leq \lambda_1\) for all \(x \in \Omega\) and \(p(x) < \lambda_1\) on some \(\Omega_0 \subset \Omega_1\) with \(|\Omega_0| > 0\), where \(\Omega_1 := \{x \in \Omega : \phi_1(x) \neq 0\}\) and \(\lambda_1 > 0\) that has an associated eigenfunction \(\phi_1\) is the first eigenvalue of \(-\Delta_\gamma\) with homogeneous Dirichlet boundary data;

(A3) \(f(x, \xi)\) is superlinear at infinity, i.e.

\[\lim_{|\xi| \to +\infty} f(x, \xi)/|\xi| = +\infty \quad \text{uniformly for all} \quad x \in \Omega;\]

(A4) There exist \(\theta \geq 1\) and \(C(x) \in L^1_+ (\Omega)\) such that \(\theta F(x, \xi) \geq F(x, s\xi) - C(x)\) for \((x, \xi) \in \Omega \times \mathbb{R}\) and \(s \in [0, 1]\), where \(F(x, \xi) = f(x, \xi) - 2F(x, \xi)\).

The condition (A4) was first introduced by L. Jeanjean [7], there are many functions which satisfy (A4), but do not satisfy the (AR) condition. An example of such function is

\[f(x, \xi) = \xi \ln(1 + |\xi|).\]

Our main result is given by the following theorem.

**Theorem 1.1.** Assume conditions (A1)-(A4) hold. Then the problem (1.1)-(1.2) has at least three nontrivial solutions.

The structure of our note is as follows: In Section 2, we give some preliminary results. In Section 3, we proved Theorem 1.1.

## 2. Preliminary results

First of all, let us collect some concepts and results of Morse theory that will be used below. For the details, we refer to [4]. Let \(X\) be a real Banach space and \(\Phi \in C^1(X, \mathbb{R})\). \(K = \{u \in X | \Phi'(u) = 0\}\) is the critical set of \(\Phi\). Let \(u \in K\) be an isolated critical point of \(\Phi\) with \(\Phi(u) = c \in \mathbb{R}\), and \(U\) be an isolated neighborhood of \(u\), i.e. \(K \cap U = \{u\}\). The group

\[C_m(\Phi, u) = H_m(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}), \quad m = 0, 1, 2, \ldots,\]

is called the \(m\)-th critical group of \(\Phi\) at \(u\), where \(\Phi^c = \{u \in X | \Phi(u) \leq c\}\).

\(H_m(\cdot, \cdot)\) is the singular relative homology group of \(\Phi\) at infinity is defined by

\[C_m(\Phi, \infty) = H_m(X, \Phi^c), \quad m = 0, 1, 2, \ldots.\]

We denote

\[P(u, t) = \sum_i \text{rank } C_i(\Phi, u)t^i, \quad P(\infty, t) = \sum_i \text{rank } C_i(\Phi, \infty)t^i.\]
Let $\alpha < \beta$ be the regular values of $\Phi$ and set
\[ P(\alpha, \beta, t) = \sum \text{rank } C_i(\Phi, \infty) t^i. \]

If $K = \{u_1, u_2, \ldots, u_k\}$, then there is a polynomial $Q(t)$ with nonnegative integer as its coefficients such that
\[ \sum_j P(u_j, t) = P(\infty, t) + (1 + t)Q(t), \quad (2.1) \]
\[ \sum_{\alpha < \Phi(u_j) < \beta} P(u_j, t) = P(\alpha, \beta, t) + (1 + t)Q(t). \quad (2.2) \]

Throughout the paper $\Omega$ denotes a bounded domain with smooth boundary in $\mathbb{R}^N, N \geq 2$. As in [11], we consider the operators of the form
\[ \Delta_\gamma := \sum_{j=1}^N \partial_{x_j} \left( \gamma_j^2 \partial_{x_j} \right), \quad \partial_{x_j} := \frac{\partial}{\partial x_j}, j = 1, 2, \ldots, N. \]

Here, the functions $\gamma_j : \mathbb{R}^N \rightarrow \mathbb{R}$ are assumed to be continuous, different from zero and of class $C^1$ in $\mathbb{R}^N \setminus \Pi$, where
\[ \Pi := \left\{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \prod_{j=1}^N x_j = 0 \right\}. \]

Moreover, we assume the following properties:
i) There exists a semigroup of dilations $\{\delta_t\}_{t > 0}$ such that
\[ \delta_t : \mathbb{R}^N \rightarrow \mathbb{R}^N, \delta_t (x_1, \ldots, x_N) = (t^{\varepsilon_1} x_1, \ldots, t^{\varepsilon_N} x_N), 1 = \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_N, \]
such that $\gamma_j$ is $\delta_t$–homogeneous of degree $\varepsilon_j - 1$, i.e.,
\[ \gamma_j (\delta_t (x)) = t^{\varepsilon_j - 1} \gamma_j (x), \forall x \in \mathbb{R}^N, \forall t > 0, j = 1, \ldots, N. \]

The number
\[ \tilde{N} := \sum_{j=1}^N \varepsilon_j \]
is called the homogeneous dimension of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t > 0}$.

ii) $\gamma_1 = 1, \gamma_j (x) = \gamma_j (x_1, x_2, \ldots, x_{j-1}), j = 2, \ldots, N.$

iii) There exists a constant $\rho \geq 0$ such that
\[ 0 \leq x_k \partial_{x_k} \gamma_j (x) \leq \rho \gamma_j (x), \forall k \in \{1, 2, \ldots, j - 1\}, \forall j = 2, \ldots, N, \]
and for every $x \in \mathbb{R}^N_+$ := \{
\[ (x_1, \ldots, x_N) \in \mathbb{R}^N : x_j \geq 0, \forall j = 1, 2, \ldots, N \}\}.

iv) Equalities $\gamma_j (x) = \gamma_j (x^*)$ $(j = 1, 2, \ldots, N)$ are satisfied for every $x \in \mathbb{R}^N$, where
\[ x^* = (|x_1|, \ldots, |x_N|) \text{ if } x = (x_1, x_2, \ldots, x_N). \]

**Definition 2.1.** By $S_{\gamma}^p (\Omega)$ $(1 \leq p < +\infty)$ we will denote the set of all functions $u \in L^p (\Omega)$ such that $\gamma_j \partial_{x_j} u \in L^p (\Omega)$ for all $j = 1, \ldots, N$. We define the norm in this space as follows
\[ \|u\|_{S_{\gamma}^p (\Omega)} = \left\{ \int_{\Omega} \left( |u|^p + \sum_{j=1}^N |\gamma_j \partial_{x_j} u|^p \right) \, dx \right\}^{\frac{1}{p}}. \]
If \( p = 2 \) we can also define the scalar product in \( S^2_\gamma(\Omega) \) as follows

\[
(u, v)_{S^2_\gamma(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{j=1}^{N} (\gamma_j \partial_{x_j} u, \gamma_j \partial_{x_j} v)_{L^2(\Omega)}.
\]

The space \( S^p_{\gamma,0}(\Omega) \) is defined as the closure of \( C^{1,0}_0(\Omega) \) in the space \( S^p_\gamma(\Omega) \).

Set

\[
\nabla_\gamma u := (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \ldots, \gamma_N \partial_{x_N} u),
\]

\[
|\nabla_\gamma u| := \left( \sum_{j=1}^{N} |\gamma_j \partial_{x_j} u|^2 \right)^{\frac{1}{2}}.
\]

From Proposition 3.2 and Theorem 3.3 in [11], we have the following embedding result.

**Proposition 2.1.** Assume that \( \tilde{N} \) \( >2 \). Then \( S^2_{\gamma,0}(\Omega) \hookrightarrow L^p(\Omega) \), where \( 1 \leq p \leq \frac{2\tilde{N}}{\tilde{N} - 1} \). Moreover, the number \( 2^*_\gamma = \frac{2\tilde{N}}{N - 2} \) is the critical Sobolev exponent of the embedding \( S^2_{\gamma,0}(\Omega) \hookrightarrow L^p(\Omega) \) and when \( 1 \leq p < 2^*_\gamma \), the embedding is compact.

We now give some examples of the \( \Delta_\gamma \)-Laplace operator. We use the following notations: we split \( \mathbb{R}^N \) into

\[
\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},
\]

and write

\[
x = (x^{(1)}, x^{(2)}, x^{(3)}), \quad x^{(i)} = (x^{(i)}_1, x^{(i)}_2, \ldots, x^{(i)}_{N_i}) \in \mathbb{R}^{N_i},
\]

\[
|x^{(i)}|^2 = \sum_{j=1}^{N_i} |x^{(i)}_j|^2, \quad i = 1, 2, 3.
\]

We denote the classical Laplace operator in \( \mathbb{R}^{N_i} \) by

\[
\Delta_{x^{(i)}} := \sum_{j=1}^{N_i} \partial_{x^{(i)}_j}^2.
\]

**Example 2.2.** Let \( \alpha \) be a real positive number. The operator

\[
\Delta_\gamma := \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha} (\Delta_{x^{(2)}} + \Delta_{x^{(3)}}),
\]

where

\[
\gamma = (\underbrace{1, 1, \ldots, 1}_{N_1 - \text{times}}, |x^{(1)}|^\alpha, \ldots, |x^{(1)}|^\alpha),
\]

is called the Grushin operator (see [6]).

**Example 2.3.** Let \( \alpha, \beta \) be nonnegative real numbers. The operator

\[
\Delta_\gamma := \Delta_{x^{(1)}} + \Delta_{x^{(2)}} + |x^{(1)}|^{2\alpha} |x^{(2)}|^{2\beta} \Delta_{x^{(3)}},
\]

where

\[
\gamma = (\underbrace{1, 1, \ldots, 1}_{(N_1 + N_2) - \text{times}}, |x^{(1)}|^\alpha |x^{(2)}|^\beta, \ldots, |x^{(1)}|^\alpha |x^{(2)}|^\beta),
\]

is called the strongly degenerate elliptic operators (see [24, 28]).
Definition 2.4. A function $u \in S_{\gamma,0}^2(\Omega)$ is called a weak solution of the problem (1.1)–(1.2) if the identity
\[
\int_{\Omega} \nabla_{\gamma} u \cdot \nabla_{\gamma} \varphi \, dx - \int_{\Omega} f(x,u) \varphi \, dx = 0,
\]
holds for every $\varphi \in C_0^\infty(\Omega)$.

Definition 2.5. Let $\mathbb{X}$ be a real Banach space with its dual space $\mathbb{X}^*$ and $\Phi \in C^1(\mathbb{X}, \mathbb{R})$. The functional $\Phi$ is said to satisfy Cerami condition at level $c \in \mathbb{R}$ ((C)_c condition for short) if for any sequence $\{x_m\}_{m=1}^\infty \subset \mathbb{X}$ with
\[
\Phi(x_m) \to c \text{ and } (1 + \|x_m\|_{\mathbb{X}}) \|\Phi'(x_m)\|_{\mathbb{X}^*} \to 0,
\]
then there exists a subsequence $\{x_{m_k}\}_{k=1}^\infty$ that converges strongly in $\mathbb{X}$. $\Phi$ satisfies the (C) condition if $\Phi$ satisfies (C)_c condition at every $c \in \mathbb{R}$.

3. Proof of the main result

First, we observe that the problem (1.1)–(1.2) has a variational structure. Indeed it is the Euler-Lagrange equation of the functional $\Phi : S_{\gamma,0}^2(\Omega) \to \mathbb{R}$ defined as follows:
\[
\Phi(u) = \frac{1}{2} \int_{\Omega} |\nabla_{\gamma} u|^2 \, dx - \int_{\Omega} F(x,u) \, dx,
\]
By the hypotheses on $f$, we can see that the functional $\Phi$ is Fréchet differentiable in $S_{\gamma,0}^2(\Omega)$ and for any $\varphi \in S_{\gamma,0}^2(\Omega)$,
\[
\langle \Phi'(u), \varphi \rangle = \int_{\Omega} \nabla_{\gamma} u \cdot \nabla_{\gamma} \varphi \, dx - \int_{\Omega} f(x,u) \varphi \, dx.
\]
Thus, critical points of $\Phi$ are solutions of problem (1.1)–(1.2).

Let
\[
f_\pm(x,\xi) = \begin{cases} 
f(x,\xi), & \xi > 0, \\
0, & \xi \leq 0;
\end{cases}
\]
\[
\Phi_\pm(u) = \frac{1}{2} \int_{\Omega} |\nabla_{\gamma} u|^2 \, dx - \int_{\Omega} F_\pm(x,u) \, dx,
\]
where $F_\pm(x,\xi) = \int_0^\xi f_\pm(x,\tau) \, d\tau$. Now, we prove the following compactness condition for $\Phi$ and $\Phi_\pm$.

Lemma 3.1. Let (A1)-(A4) be satisfied. Then the functionals $\Phi$ and $\Phi_\pm$ satisfies the (C) condition on $S_{\gamma,0}^2(\Omega)$.

Proof. We only give the proof for $\Phi_+$, the cases of $\Phi$ and $\Phi_-$ are similar. Let $\{u_n\}_{n=1}^\infty \subset S_{\gamma,0}^2(\Omega)$ be a sequence such that
\[
\Phi_+(u_n) \to c, \quad \left(1 + \|u_n\|_{S_{\gamma,0}^2(\Omega)}\right) \|\Phi'_+(u_n)\|_{(S_{\gamma,0}^2(\Omega))^*} \to 0, \quad \text{as } n \to \infty. \tag{3.1}
\]
The proof of this lemma, we divide two steps:

Step 1. We first prove that $\{u_n\}_{n=1}^\infty$ is bounded in $S_{\gamma,0}^2(\Omega)$. Let $u_+ = \max\{u_n,0\}$, $u_- = \min\{u_n,0\}$. From (3.1), we obtain
\[
\|\Phi'_+(u_n)\|_{(S_{\gamma,0}^2(\Omega))^*} \leq \epsilon_n \|\varphi\|_{S_{\gamma,0}^2(\Omega)} \quad \text{for any } \varphi \in S_{\gamma,0}^2(\Omega), \tag{3.2}
\]
where $\epsilon_n \to 0$ as $n \to \infty$, then the boundedness of $u_-^-$ can be directly obtained. For the case of $u_+^+$, by contradiction, we assume that $\|u_n^+\|_{S_{\gamma,0}^2(\Omega)} \to \infty$ as $n \to \infty$. Let $v_n = \|u_n^+\|_{S_{\gamma,0}^2(\Omega)}^{-1} u_n^+$, then
\[ \| v_n \|_{S^2_{\gamma,0}(\Omega)} = 1. \] By Proposition 2.1, up to a subsequence, we have
\[ v_n \rightharpoonup v \quad \text{weakly in } S^2_{\gamma,0}(\Omega) \text{ as } n \to \infty, \]
\[ v_n \to v \quad \text{strongly in } L^q(\Omega) \text{ as } n \to \infty, \]
\[ v_n \to v \quad \text{a.e. in } \Omega \text{ as } n \to \infty. \]

**Case 1.** If \( v \neq 0 \) then the Lebesgue measure of \( \Omega_0 = \{ x \in \Omega : v(x) \neq 0 \} \) is positive. Using (3.1), we obtain
\[ \langle \Phi'_+(u_n), u^+_n \rangle = o(1), \]
which implies that
\[ \int_{\Omega} \frac{f_+(x, u^+_n)}{\| u^+_n \|^2_{S^2_{\gamma,0}(\Omega)}} | u^+_n |^2 v_n^2 dx = 1 + o(1). \] (3.3)

By (A3), there is a constant \( M > 0 \) such that
\[ f_+(x, u^+_n) > 0, \quad \text{as } |u_n| > M, \]
then we have
\[ \int_{\Omega_0} \frac{f_+(x, u^+_n)}{(u^+_n)^2} | v_n |^2 dx \geq -C. \] (3.4)

On the other hand, for \( x \in \Omega_0, u^+_n \to \infty \) as \( n \to \infty \). Then by the Fatou’s lemma and (A3) we have
\[ \int_{\Omega_0} \frac{f_+(x, u^+_n)}{(u^+_n)^2} | v_n |^2 dx \to \infty, \quad \text{as } n \to \infty. \]

Combining this with (3.4) gives
\[ \int_{\Omega} \frac{f_+(x, u^+_n)}{(u^+_n)^2} | v_n |^2 dx \to \infty, \quad \text{as } n \to \infty. \] (3.5)

This contradicts (3.3). Then this case is impossible.

**Case 2.** If \( v \equiv 0 \) then for any \( n \in \mathbb{N} \) there exists \( t_n \in [0, 1] \) such that
\[ \Phi_+(t_n u^+_n) = \max_{t \in [0, 1]} \Phi_+(t u^+_n). \]

For any \( R > 0 \), we assume that \( w_n = 2\sqrt{R} v_n \). Then \( w_n \to 0 \) in \( L^q(\mathbb{R}^N) \). So from conditions (A1) and (A2), for every \( \epsilon > 0 \), we can find a constant \( C(\epsilon) > 0 \) such that
\[ F(x, w_n) \leq C(\epsilon)(w_n)^2 + \epsilon(w_n)^{2^*_q}, \] (3.6)

which implies
\[ \lim_{n \to \infty} \int_{\Omega} F_+(x, w_n) dx = 0. \] (3.7)

Since \( 2\sqrt{R} \| u^+_n \|_{S^2_{\gamma,0}(\Omega)} = 0 \) for \( n \) large enough, by (3.7) we obtain
\[ \Phi_+(t_n u^+_n) \geq \Phi_+(w_n) = 2R - \int_{\Omega} F_+(x, w_n) dx \geq R, \]
which implies
\[ \Phi_+(t_n u^+_n) \to \infty, \quad \text{as } n \to \infty. \] (3.8)
From \( \Phi_+(0) = 0 \) and \( \Phi_+(u_n^+) \to c \) we have \( t_n \in (0, 1) \), then
\[
\langle \Phi'_+(t_n u_n^+), t_n u_n^+ \rangle = t_n \frac{d}{dt} \big|_{t=t_n} \Phi_+(t u_n) = 0.
\]
Then, from (A4) it follows that
\[
\frac{1}{\theta} \Phi_+(t_n u_n^+) = \frac{1}{\theta} \left( \Phi_+(t_n u_n^+) - \frac{1}{2} \langle \Phi'_+(t_n u_n^+), t_n u_n^+ \rangle \right)
= \frac{1}{2} \int_\Omega \mathcal{F}(x, t_n u_n^+) dx
\leq \frac{1}{2} \int_\Omega \mathcal{F}(x, u_n^+) dx + \frac{1}{2\theta} \int_\Omega C(x) dx
= \Phi_+(u_n^+) - \frac{1}{2} \langle \Phi'_+(u_n^+), u_n^+ \rangle + c \to C.
\]
This contradicts that \( \Phi_+(t_n u_n^+) \to \infty \). Hence \( \{u_n\}_{n=1}^\infty \) is bounded; that is, there exists a positive constant \( M \) such that
\[
\|u_n\|_{S_{\gamma,0}^2(\Omega)} \leq M, \quad \text{for all } n \in \mathbb{N}.
\]

**Step 2.** We prove \( \{u_n\}_{n=1}^\infty \) has a convergent subsequence. In fact, we can suppose that
\[
u_n \rightharpoonup u \quad \text{weakly in } S_{\gamma,0}^2(\Omega) \text{ as } n \to \infty,
\]
\[
u_n \to u \quad \text{strongly in } L^q(\Omega) \text{ as } n \to \infty,
\]
\[
u_n \to u \quad \text{a.e. in } \Omega \text{ as } n \to \infty.
\]
Now, since \( \Omega \) is a bounded set, for every \( \epsilon > 0 \), we can find a constant \( C(\epsilon) > 0 \) such that
\[
f_+(x,s) \leq C(\epsilon) + \epsilon |s|^{2\gamma - 1}, \quad \forall (x,s) \in \Omega \times \mathbb{R},
\]
then
\[
\left| \int_\Omega f_+(x, u_n)(u_n - u) dx \right|
\leq C(\epsilon) \int_\Omega |u_n - u| dx + \epsilon \int_\Omega |u_n - u|^{2\gamma - 1} dx
\leq C(\epsilon) \int_\Omega |u_n - u| dx + \epsilon \left( \int_\Omega (|u_n|^{2\gamma - 1})^{\frac{2\gamma - 1}{\gamma}} dx \right)^{\frac{\gamma}{2\gamma - 1}} \left( \int_\Omega |u_n - u|^{2\gamma} dx \right)^{\frac{1}{2\gamma}}
\leq C(\epsilon) \int_\Omega |u_n - u| dx + \epsilon C(\Omega).
\]
Similarly, since \( u_n \to u \) in \( S_{\gamma,0}^2(\Omega) \), it follows that \( \int_\Omega |u_n - u| dx \to 0 \). Since \( \epsilon > 0 \) is arbitrary, we can conclude that
\[
\int_\Omega (f_+(x, u_n) - f_+(x, u))(u_n - u) dx \to 0 \quad \text{as } n \to \infty.
\]
By (3.9), we have
\[
\langle \Phi'_+(u_n) - \Phi'_+(u), (u_n - u) \rangle \to 0 \quad \text{as } n \to \infty.
\]
From (3.9) and (3.10), we obtain \( \|u_n\|_{S_{\gamma,0}^2(\Omega)} \to \|u\|_{S_{\gamma,0}^2(\Omega)} \), as \( n \to \infty \). Thus we have
\[
\|u_n - u\|_{S_{\gamma,0}^2(\Omega)} \to 0, \quad \text{as } n \to \infty,
\]
which means that \( \Phi_+ \) satisfies condition (C).
Proof. We only give the proof of $\Phi_+$; the others are similar. Let $S = \{u \in S^2_{γ,0}(Ω) : \|u\|_{S^2_{γ,0}(Ω)} = 1, u^+ \neq \emptyset\}$ and $B^\infty = \{u \in S^2_{γ,0}(Ω) : \|u\|_{S^2_{γ,0}(Ω)} \leq 1\}$. By (A3), for any $M > 0$ there exists $c > 0$, such that $F(x, t) \geq Mt^2 - c$, for $(x, t) \in Ω \times R$, which implies $\Phi_+(tu) \to -\infty$, as $t \to +\infty$, for any $u \in S$. Using (A4), we have

\[
f_+(x, t) t - 2F(x, t) \geq -\frac{C(x)}{θ}, \quad \text{for} \quad (x, t) \in Ω \times R.
\]  

Choose

\[
a < \min\left\{ \inf_{u \in B^\infty} \Phi_+(u), -\frac{C_*}{2θ}\right\},
\]

where $C_* = \int_Ω C(x)dx$. Then for any $u \in S$, there exists $t > 1$ such that $\Phi_+(tu) \leq a$, that is

\[
\Phi_+(tu) = \frac{t^2}{2} - \int_Ω F_+(x, tu)dx \leq a,
\]

which (3.11) implies

\[
\frac{d}{dt}\Phi_+(tu) = t - \int_Ω f_+(x, tu)u \leq \frac{1}{t}(2a + \frac{C_*}{θ}) < 0.
\]

Therefore, by the implicit function theorem, there exists a unique $T \in C(S, R)$ such that

\[
\Phi_+(T(u))u = a, \quad \text{for} \quad u \in S.
\]

Let $S_1 = \{u \in S^2_{γ,0}(Ω) : \|u\|_{S^2_{γ,0}(Ω)} \geq 1, u^+ \neq \emptyset\}$. We construct a strong deformation retract $τ : [0, 1] \times S_1 \to S_1$ which satisfies $τ(s, u) = (1 - s)u + sT(\frac{u}{\|u\|_{S^2_{γ,0}(Ω)}}) \frac{u}{\|u\|_{S^2_{γ,0}(Ω)}}$ if $\Phi_+(u) \geq a$ and $τ(s, u) = u$ if $\Phi_+(u) < a$. Hence, It follows from the construction of $τ$ that $Φ^a_+$ is a strong deformation retract of $S_1$, which is homotopy equivalent to the set $S$. By the homotopy invariance of homology group, we have

\[
C_m(Φ_+, Ω) = H_m(S^2_{γ,0}(Ω), Φ^a_+),
\]

\[
≡ H_m(S^2_{γ,0}(Ω), S)
\]

\[
≡ H_m(S^2_{γ,0}(Ω), S^2_{γ,0}(Ω) \setminus \{0\})
\]

\[
= 0.
\]

Proof of Theorem 1.1. By Lemma 3.1, we know that $Φ$ and $Φ_\pm$ satisfy the (C) condition. By conditions (A1) and (A2), we can easily prove that 0 is a local minimum of $Φ$ and $Φ_\pm$. So, we have

\[
C_m(Φ, 0) = C_m(Φ_\pm, 0) = δ_{m,0}G.
\]  

(3.12)

Using the mountain pass theorem in [21], we obtain $Φ_\pm$ ($Φ_-$) has a critical point $u_+ > 0$ ($u_- < 0$), and $u_\pm$ are also the nontrivial critical points of the functional $Φ$. Without loss of generality, we assume that $u_\pm$ are isolated and the only nontrivial critical points of the functional $Φ$. Now we claim that

\[
C_m(Φ_\pm, u_\pm) = δ_{m,1}G.
\]  

(3.13)

Indeed, using the methods of [9], we let $Φ_+(u_+) = c > 0$. It follows from the homology exact sequence of the triple $Φ^+_1 \subset Φ^+_\pm \subset S^2_{γ,0}(Ω)$, we have

\[
\cdots \to H_m(S^2_{γ,0}(Ω), Φ^+_1) \to H_m(S^2_{γ,0}(Ω), Φ^+_\pm) \to H_{m-1}(Φ^+_\pm, Φ^+_1) \to \cdots
\]

\[
\to H_{m-1}(S^2_{γ,0}(Ω), Φ^+_\pm) \to \cdots
\]

(3.14)
where $A < 0$ is a constant. Since 0 is the only critical point of $\Phi_+$ in the set $\Phi^+_m$, by (3.12), we obtain
\begin{equation}
H_m(\Phi^+_m, \Phi^+_m) = C_m(\Phi^+_+, 0) = \delta_{m,0}G.
\end{equation}

Similarly, since $u_+$ is the only critical point of $\Phi_+$ in the set $\{u \in S^2_{(\gamma,0)}(\Omega) | \Phi_+(u) \geq \frac{\gamma}{2}\}$, we have
\begin{equation}
H_m(S^2_{(\gamma,0)}(\Omega), \Phi^+_m) = C_m(\Phi_+, u_1), \quad m = 0, 1, 2, \ldots.
\end{equation}

From Lemma 3.2, we have
\begin{equation}
H_m(S^2_{(\gamma,0)}(\Omega), \Phi^+_m) = C_m(\Phi_+, \infty) = 0, \quad m = 0, 1, 2, \ldots.
\end{equation}

From (3.14) to (3.17), we deduce that $C_m(\Phi_+, u_+) = C_{m-1}(\Phi_+, 0) = \delta_{m,1}G$.

The case for $u_-$ is similar, that is $C_m(\Phi_-, u_-) = C_{m-1}(\Phi_-, 0) = \delta_{m,1}G$.

Hence $C_m(\Phi, u_\pm) = \delta_{m,1}G$.

The Morse equality (2.1) with $t = -1$ implies that $(-1)^0 + (-1)^1 + (-1)^1 = 0$, which is a contradiction. Then the problem (1.1)–(1.2) has at least three nontrivial solutions.

References


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