Existence of a Renormalized Solution of Nonlinear Parabolic Equations With General Measure Data

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ABSTRACT: In this paper we prove the existence of a renormalized solution for nonlinear parabolic equations of the type:
\[ \frac{\partial b(x,u)}{\partial t} - \text{div}\left( a(x,t,\nabla u) \right) = \mu \quad \text{in } \Omega \times (0,T), \]
where the right hand side is a general measure, \( b(x,u) \) is an unbounded function of \( u \) and \( -\text{div}(a(x,t,\nabla u)) \) is a Leray–Lions type operator with growth \( |\nabla u|^{p-1} \) in \( \nabla u \).

Key Words: Nonlinear parabolic equations, Renormalized solutions, General measure.

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1. Introduction

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \), \( (N \geq 1) \), \( T \) is a positive real number, and let \( Q := \Omega \times (0,T) \), \( p > 1 \). We will consider the following nonlinear parabolic problem
\[ \frac{\partial b(x,u)}{\partial t} - \text{div}\left( a(x,t,\nabla u) \right) = \mu \quad \text{in } Q, \]
\[ b(x,u)(t = 0) = b(x,u_0) \quad \text{in } \Omega, \]
\[ u = 0 \quad \text{on } \partial \Omega \times (0,T). \]

In Problem (1.1)-(1.3) the framework is the following: the data \( \mu \) is a bounded Radon measure on \( Q \), the function \( b(x,u_0) \) belongs to \( L^1(\Omega) \).
The operator \( -\text{div}(a(x,t,\nabla u)) \) is a Leray–Lions operator which is coercive and which grows like \( |\nabla u|^{p-1} \) with respect to \( \nabla u \), (see assumptions (3.4), (3.5) and (3.6) of Section 3).

In the case where \( b(x,u) = u \), and the right hand side is a bounded measure, the existence of a distributional solution was proved in [16], but due the lack of regularity of the solutions, the distributional formulation is not strong enough to provide uniqueness (see [41] and [35] for a counter example in the elliptic case). To overcome this difficulty the notion of renormalized solutions firstly introduced by DiPerna and Lions [23] for the study of Boltzmann equation was adapted to the parabolic and elliptic equations with \( L^1 \) data (see [4,6,7,30,31,34]). The equivalent notion of entropy solution has been developed independently by [3] for the study of nonlinear elliptic problems and by [37] in the parabolic case. Both renormalized and entropy solutions provide a convenient framework to deal with elliptic or parabolic equations with \( L^1 \) data. A large number of papers was then devoted to the study the existence of renormalized (or entropy) solution of parabolic problems with rough data under various
assumptions and different contexts: in addition to the references already mentioned see, among others, [1,2,9,10,11,12,13,18,19,32,40].

Concerning the datum \( \mu \), the existence and uniqueness of renormalized solution of (1.1)-(1.3) have proved in [39] in the case where \( b(x,u) = u \), \( u_0 \in L^1(\Omega) \) and for every measure \( \mu \) which does not charge the sets of zero \( p \)-capacity, the so-called diffuse measures or soft measures (see Section 2 for the definitions of \( p \)-capacity and diffuse measure). The importance of the measures not charging sets of null \( p \)-capacity was first observed in the stationary case in [15], and developed in the evolution case in [39].

For \( \mu \in \mathcal{M}_0(\Omega) \), \( b(x,u) = b(u) \) and \( u_0 \in L^1(\Omega) \), the existence and uniqueness of renormalized solution have been proved in [12]. In the case where \( \mu \in \mathcal{M}_0(\Omega) \) and with the parabolic term on \( b(x,u) \), the existence of renormalized solution of problem (1.1)-(1.3) was proved in [38]. For \( \mu \in \mathcal{M}(\Omega) \) (the space of all bounded Radon measures on \( \Omega \)), \( b(x,u) = u \) and \( u_0 \in L^1(\Omega) \), the existence of renormalized solution was proved in [21] for elliptic case and [33] for parabolic case. Our goal in this paper is to extend the approach in [7] (see also [12], [38]) to the general measure data.

The paper is organized as follows as follows. In Section 2 we give some preliminaries and, in particular, we provide the definition of parabolic capacity and some basic properties. Section 3 is devoted to specify the assumptions on \( b, a, u_0 \) and \( \mu \) and to give the definition of renormalized solution of (1.1)-(1.3) and see how the definition of renormalized solution does not depend on the decomposition (not uniquely determined) of the regular part of \( \mu \) we mentioned above and to the statement of standard approximation argument we will use later. In Section 4 we establish (Theorem 4.1) the existence of such a solution.

## 2. Preliminaries on parabolic capacity

We recall the notion of \( p \)-capacity associated to our problem. Let \( Q = \Omega \times (0, T) \) for any fixed \( T > 0 \) and let us recall that \( V = W_0^{1,p}(\Omega) \cap L^2(\Omega) \), endowed with its natural norm \( \| \cdot \|_{W_0^{1,p}(\Omega)} + \| \cdot \|_{L^2(\Omega)} \) and

\[
W = \left\{ u \in L^p(0,T;V), u_\tau \in L^{p'}(0,T;V') \right\},
\]

endowed with its natural norm \( \| \cdot \|_{L^p(0,T;V)} + \| \cdot \|_{L^{p'}(0,T;V')} \); remark that \( W \) is continuously embedded in \( C([0,T],L^2(\Omega)) \), and if \( 1 < p < \infty \), then \( C^\infty_c(\Omega \times [0,T]) \) is dense in \( W \).

Let \( U \subseteq Q \) be an open set, we define the parabolic \( p \)-capacity of \( U \) as

\[
cap_p(U) = \inf \left\{ \| u \|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q \right\},
\]

where as usual we set \( \inf \emptyset = +\infty \), then for any Borel set \( B \subseteq Q \) we define

\[
cap_p(B) = \inf \left\{ \cap_p(U) : U \text{ open set of } Q, B \subseteq U \right\}.
\]

We define the space \( S \) by

\[
S = \left\{ u \in L^p(0,T;W_0^{1,p}(\Omega)), u_\tau \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(\Omega) \right\},
\]

endowed with its natural norm \( \| \cdot \|_{L^p(0,T;W_0^{1,p}(\Omega))} + \| \cdot \|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + L^1(\Omega) \).

We will denote by \( \mathcal{M}(Q) \) the set of all Radon measures with bounded total variation on \( Q \), while \( \mathcal{M}_0(Q) \) is the set of all measures with bounded total variation over \( Q \) that do not charge the sets of zero \( p \)-capacity, that is if \( \mu \in \mathcal{M}_0(Q) \), then \( \mu(E) = 0 \), for all \( E \subseteq Q \) such that \( \cap_p(E) = 0 \).

**Theorem 2.1.** Let \( \mu \) be a bounded measure in \( Q \). If \( \mu \in \mathcal{M}_0(Q) \) then there exists \( (f,g_1,g_2) \) such that \( f \in L^1(Q), g_1 \in L^p(0,T;W^{-1,p'}(\Omega)), g_2 \in L^p(0,T;V) \) and

\[
\int_Q \phi \, d\mu = \int_Q f \phi \, dx \, dt + \int_0^T \langle g_1, \phi \rangle \, dt - \int_0^T \langle g_2, \phi \rangle \, dt \quad \phi \in C_c^\infty(\Omega \times [0,T]).
\]

Such a triplet \( (f,g_1,g_2) \) will called the decomposition of \( \mu \).
Proof. See [39].

By a well known decomposition result (see for instance [27]), every \( \mu \) in \( \mathcal{M}(Q) \), can be written as a sum (uniquely determined) of its absolutely continuous part \( \mu_a \) with respect to p-capacity and its singular part \( \mu_s \) concentrated on a set \( E \) of zero p-capacity; we will say that \( \mu_s \perp \text{cap}_p \). Hence, if \( \mu \in \mathcal{M}(Q) \), thanks to Theorem 2.1, we have

\[
\mu = f - \text{div}(G) + g_t + \mu^+_s - \mu^-_s,
\]

in the sense of distributions, for \( f \in L^1(Q) \), \( G \in (L^p(Q))^N \), \( g \in L^p(0,T;V) \), where \( \mu^+_s \) and \( \mu^-_s \) are respectively the positive and the negative part of \( \mu_s \); note that the decomposition of the absolutely continuous part of \( \mu \) according to Theorem 2.1 is not uniquely determined. Let us state the following result that will be very useful in the sequel; its proof relies on an easy application of Egorov and Dunford-Pettis theorems.

**Proposition 2.1.** Let \( \rho_\varepsilon \) be a sequence of \( L^1(Q) \) functions that converges to \( \rho \) weakly in \( L^1(Q) \), and let \( \sigma_\varepsilon \) be sequence of functions in \( L^\infty(Q) \) that is bounded in \( L^\infty(Q) \) and converges to \( \sigma \) almost everywhere on \( Q \). Then

\[
\lim_{\varepsilon \to 0} \int_Q \rho_\varepsilon \sigma_\varepsilon \, dx dt = \int_Q \rho \sigma \, dx dt.
\]

Here are some notations we will use throughout this paper. For any non negative real number \( k \) we denote by \( T_k(r) = \min(k,\max(r,-k)) \) the truncation function at level \( k \). By \( \langle ., . \rangle \) we mean the duality between suitable spaces in which functions are involved, in particular we will consider both the duality between \( W^{1,p}_0(\Omega) \) and \( W^{-1,p'}(\Omega) \) and the duality between \( W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) and \( W^{-1,p'}(\Omega) + L^1(\Omega) \).

3. Assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true: \( \Omega \) is a bounded open set on \( \mathbb{R}^N \) (\( N \geq 1 \)), \( T > 0 \) is given and we set \( Q = \Omega \times (0,T) \)

\[
b, \frac{\partial b}{\partial s} : \Omega \times \mathbb{R} \to \mathbb{R} \text{ and } \nabla_x b : \Omega \times \mathbb{R} \to \mathbb{R}^N, \text{ are Carathéodory functions}
\]

such that for almost every \( x \in \Omega \), \( b(x,.) \) is a strictly increasing \( C^1 \)-function with \( b(x,0) = 0 \). For all \( s \in \mathbb{R} \), \( b(.,s) \) is in \( W^{1,p}(\Omega) \), and there exist \( \gamma, \Lambda > 0 \) such that

\[
\gamma \leq \frac{\partial b(x,s)}{\partial s} \leq \Lambda,
\]

for almost every \( x \in \Omega \), for every \( s \in \mathbb{R} \). There exists a function \( B \) in \( L^p(\Omega) \) such that

\[
\left| \nabla_x b(x,s) \right| \leq B(x),
\]

for almost every \( x \in \Omega \), for every \( s \in \mathbb{R} \).

\[
a : Q \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a Carathéodory function}
\]

\[
a(x,t,\xi) . \xi \geq \alpha |\xi|^p,
\]

for almost every \( (x,t) \in Q \), for every \( \xi \in \mathbb{R}^N \), where \( \alpha > 0 \) given real number.

\[
|a(x,t,\xi)| \leq \beta (L(x,t) + |\xi|^{p-1}),
\]
for almost every \((x, t) \in Q\), for every \(\xi \in \mathbb{R}^N\), where \(\beta > 0\) given real number, \(L\) is a non negative function in \(L^p(Q)\).

\[
[a(x, t, \xi) - a(x, t, \xi^\prime)][\xi - \xi^\prime] > 0,
\]

for any \((\xi, \xi^\prime) \in \mathbb{R}^{2N}\) with \(\xi \neq \xi^\prime\) and for almost every \((x, t) \in Q\).

\[
\mu \in M(Q),
\]

\[
u_0 \in L^1(\Omega).
\]

To simplify notation, let us also define \(v = b(x, u) - g\), the definition of a renormalized solution for Problem (1.1)-(1.3) is given below.

**Definition 3.1.** A measurable function \(u\) defined on \(Q\) is a renormalized solution of Problem (1.1)-(1.3) if

\[
T_k(v) \in L^p(0, T; W^{1, p}_0(\Omega)) \quad \forall k \geq 0, \quad v \in L^\infty(0, T; L^1(\Omega)),
\]

and, for every function \(S\) in \(W^{2, \infty}(\mathbb{R})\), which is piecewise \(C^1\) and such that \(S'\) has a compact support and \(S(0) = 0\), we have

\[
S(v)_t - \operatorname{div}\left(S'(v) a(x, t, \nabla u)\right) + S''(v) a(x, t, \nabla u) \nabla v = f S'(v) - \operatorname{div}\left(G S'(v)\right) + G S''(v) \nabla v \quad \text{in} \quad D'(Q),
\]

\[
S(v)(t = 0) = S(b(x, u_0)) \quad \text{in} \quad L^1(\Omega).
\]

For every \(\psi \in C(\overline{Q})\), we have

\[
\lim_{n \to \infty} \frac{1}{n} \int_{\{x, t \in Q : n \leq v < 2n\}} a(x, t, \nabla u) \nabla v \psi \, dx \, dt = \int_Q \psi \, d\mu^+_a,
\]

\[
\lim_{n \to \infty} \frac{1}{n} \int_{\{x, t \in Q : -2n < v \leq -n\}} a(x, t, \nabla u) \nabla v \psi \, dx \, dt = \int_Q \psi \, d\mu^-_a.
\]

**Remark 3.2.** Note that, all terms in (3.11) are well defined, indeed, let \(k > 0\) such that \(\operatorname{supp}(S') \subset [-k, k]\), we have \(\nabla S(v) = S'(T_k(v)) \nabla T_k(v) \in (L^p(\Omega))^N\), then \(S(v) \in L^p(0, T; W^{1, p}_0(\Omega))\). The term \(S'(v) a(x, t, \nabla u)\) identifies with

\[
S'(T_k(v)) a\left(x, t, \left(\frac{\partial b(x, u)}{\partial s}\right)^{-1} \left(\nabla T_k(v) + \nabla g - \nabla_x b(x, u)\right)\right)
\]

a.e. in \(Q\). Using assumptions (3.2) and (3.6) we have

\[
\left|S'(v) a(x, t, \nabla u)\right| \leq \beta \|S'\|_{L^\infty(\mathbb{R})} \left[L(x, t) + \gamma^{-p-1} \left|\nabla T_k(v) + \nabla g - \nabla_x b(x, u)\right|^{p-1}\right] \quad \text{a.e. in} \quad Q,
\]

and by (3.3) and (3.10) we deduce that \(S'(v) a(x, t, \nabla u) \in (L^p(Q))^N\).

The term \(S''(v) a(x, t, \nabla u) \nabla v\) identifies with

\[
S''(T_k(v)) a\left(x, t, \left(\frac{\partial b(x, u)}{\partial s}\right)^{-1} \left(\nabla T_k(v) + \nabla g - \nabla_x b(x, u)\right)\right) \nabla T_k(v) \quad \text{a.e. in} \quad Q.
\]

In view of (3.2), (3.3), (3.10), (3.15) and by Hölder inequality we conclude that \(S''(v) a(x, t, \nabla u) \nabla v\) in \(L^1(Q)\).
Proposition 3.1. Let \( v = b(x, u) - g \) be a renormalized solution of problem (1.1)-(1.3). Then, for every \( k > 0 \), we have
\[
\int_Q |\nabla T_k(v)|^p \, dx \, dt \leq C(k + 1),
\]
(3.16)
where \( C \) is a positive constant not depending on \( k \).

Proof. Using assumptions (3.2) and (3.3), following the same arguments as in [33], yield (3.16). \( \square \)

Here, we give two results which show that the renormalized solution does not depend on the decomposition of the regular part of \( \mu \).

Lemma 3.3. Let \( \mu_0 \in \mathcal{M}_0(\Omega) \), and let \( (f, g_1, g_2) \) and \( (\overline{f}, \overline{g}_1, \overline{g}_2) \) to be two different decomposition of \( \mu \) according to Theorem 2.1. Then we have \( (g_2 - \overline{g}_2)_t = f - f + g_1 - \overline{g}_1 \in \text{distribution sense}, g_2 - \overline{g}_2 \in C([0, T], L^1(\Omega)) \) and \( (g_2 - \overline{g}_2)(0) = 0 \).

Proof. See [39], Lemma 2.29. \( \square \)

The following result shows that the definition of renormalized solution does not depend on the decomposition of the absolutely continuous part of \( \mu \) under the condition of bounded perturbations of time derivative part of \( \mu_0 \), and thanks to the estimate (3.16).

Proposition 3.2. Let \( u \) be a renormalized solution of problem (1.1)-(1.3). Then, \( u \) satisfies the definition 3.1 for every decomposition \( (f, g_1, g_2) \) such that \( g_2 - \overline{g}_2 \in L^p(0, T; W^{1,p}_0(\Omega)) \cap L^\infty(\Omega) \).

Proof. See [33], Proposition 3 and Remark 6. \( \square \)

4. Existence result

This section is devoted to establish the following existence theorem.

Theorem 4.1. Under assumptions (3.1)-(3.9), there exists at least a renormalized solution \( u \) of the Problem (1.1)-(1.3).

Proof. We will obtain the existence result by an approximation process, we approximate the measure \( \mu \in \mathcal{M}(\Omega) \) by a sequence defined by
\[
\mu^\varepsilon = f^\varepsilon - \text{div}(G^\varepsilon) + \frac{\partial g^\varepsilon}{\partial t} + \lambda^\varepsilon_+ - \lambda^\varepsilon_-
\]
(4.1)
where \( f^\varepsilon \in C^\infty_c(\Omega) \) is a sequence of functions which converges to \( f \) weakly in \( L^1(\Omega) \), \( G^\varepsilon \in (C^\infty_c(\Omega))^N \) is a sequence of functions which converges to \( G \) strongly in \( (L^p(\Omega))^N \), \( g^\varepsilon \in C^\infty_c(\Omega) \) is a sequence of functions which converges to \( g \) strongly in \( L^p(0, T; W^{1,p}_0(\Omega)) \), and \( \lambda^\varepsilon_+ \in C^\infty_c(\Omega) \) (respectively \( \lambda^\varepsilon_- \)) is a sequence of non negatives functions that converges to \( \mu^\varepsilon_+ \) (respectively \( \mu^\varepsilon_- \)) in the narrow topology of measures. Moreover let \( u_0^\varepsilon \in C^\infty_c(\Omega) \) such that
\[
u^\varepsilon = b(x, u_0^\varepsilon) \to b(x, u_0) \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0.
\]
(4.2)
We also assume
\[ \|\mu^\varepsilon\|_{L^1(Q)} \leq C|\mu|_{M(Q)} \quad \text{and} \quad \|b(x, u_0^\varepsilon)\|_{L^1(Q)} \leq C\|b(x, u_0)\|_{L^1(Q)}. \]

Let us now consider the following regularized problem:
\[ u^\varepsilon \in L^p(0, T; W_0^{1, p}(\Omega)), \]
\[ \int_0^T \langle \frac{\partial u^\varepsilon}{\partial t}, \varphi \rangle \, dt + \int_Q a(x, t, \nabla u^\varepsilon) \nabla \varphi \, dx \, dt = \int_Q f^\varepsilon \varphi \, dx \, dt + \int_Q G^\varepsilon \nabla \varphi \, dx \, dt + \int_Q \varphi \, d\lambda^+_\varepsilon - \int_Q \varphi \, d\lambda^-_\varepsilon \quad (4.4) \]
\[ \forall \varphi \in L^p(0, T; W_0^{1, p}(\Omega)) \cap L^\infty(Q), \]
\[ b(x, u^\varepsilon)(t = 0) = b(x, u_0) \text{ in } \Omega, \quad (4.5) \]
where \( v^\varepsilon = b(x, u^\varepsilon) - g^\varepsilon. \)

As a consequence, proving existence of a weak solution \( u^\varepsilon \in L^p(0, T; W_0^{1, p}(\Omega)) \) of (4.3)-(4.5) is an easy task (see [5] and [28]).
Now we prove the following proposition which gives some compactness results.

**Proposition 4.1.** Let \( u^\varepsilon \) and \( v^\varepsilon \) be defined as before. Then
\[ \|u^\varepsilon\|_{L^{\infty}(0, T; L^1(\Omega))} \leq C, \quad (4.6) \]
\[ \int_Q |\nabla T_k(u^\varepsilon)|^p \, dx \, dt \leq Ck, \quad (4.7) \]
\[ u^\varepsilon \text{ is bounded in } L^q(0, T; W_0^{1, q}(\Omega)) \quad \forall \ 1 < q < p - \frac{N}{N + 1}, \quad (4.8) \]
\[ \|u^\varepsilon\|_{L^{\infty}(0, T; L^1(\Omega))} \leq C, \quad (4.9) \]
\[ \int_Q |\nabla T_k(v^\varepsilon)|^p \, dx \, dt \leq C(k + 1), \quad (4.10) \]
and, up to a subsequence, for any \( k > 0 \) we have
\[ u^\varepsilon \to u \ a.e. \ on \ Q \ weakly \ in \ L^q(0, T; W_0^{1, q}(\Omega)) \text{ and strongly in } L^1(Q), \quad (4.11) \]
\[ v^\varepsilon \to v \ a.e. \ on \ Q \ weakly \ in \ L^q(0, T; W_0^{1, q}(\Omega)) \text{ and strongly in } L^1(Q), \quad (4.12) \]
\[ T_k(u^\varepsilon) \to T_k(u) \ weakly \ in \ L^p(0, T; W_0^{1, p}(\Omega)) \text{ and } a.e. \ on \ Q, \quad (4.13) \]
\[ a(x, t, \nabla u^\varepsilon)\chi_{\{|\nabla u^\varepsilon| \leq k\}} \to \sigma_k \text{ in } (L^p(\Omega))^N, \quad (4.14) \]
\[ T_k(v^\varepsilon) \to T_k(v) \ weakly \ in \ L^p(0, T; W_0^{1, p}(\Omega)) \text{ and } a.e. \ on \ Q, \quad (4.15) \]
\[ \nabla_x b(x, u^\varepsilon) \to \nabla_x b(x, u) \text{ strongly in } (L^p(\Omega))^N. \quad (4.16) \]
Proof. We prove (4.6) and (4.7), using \( T_k(u^\varepsilon) \) as a test function in (4.4) and we integrate in \([0,t]\) we get
\[
\int_{\Omega} B_k(x, u^\varepsilon)(t) \, dx + \int_{0}^{t} \int_{\Omega} a(x, t, \nabla u^\varepsilon) \nabla T_k(u^\varepsilon) \, dx \, ds = \int_{0}^{t} \int_{\Omega} \mu^\varepsilon T_k(u^\varepsilon) \, dx \, ds + \int_{\Omega} B_k(x, u_0^\varepsilon) \, dx,
\]
for almost every \( t \in (0, T) \), and where \( B_k(x, s) = \int_{0}^{s} T_k(r) \frac{\partial b(x, r)}{\partial r} \, dr. \)

Using assumption (3.5) and since \( B_k(x, u^\varepsilon) \geq 0 \), (4.7) derives from (4.17). By (3.2) we have
\[
B_1(x, s) \geq \gamma \int_{0}^{s} T_1(r) \, dr \quad \forall s \in \mathbb{R},
\]
and since \( \int_{0}^{s} T_1(r) \, dr \geq |s| - 1 \quad \forall s \in \mathbb{R}, \)
we obtain
\[
\int_{\Omega} |u^\varepsilon(t)| \, dx \leq \frac{1}{1}(\| b(x, u_0^\varepsilon) \|_{L^1(\Omega)} + |\mu|_{\mathcal{M}(Q)}) + \text{meas}(\Omega).
\]
Hence \( u^\varepsilon \) is bounded in \( L^\infty(0, T; L^1(\Omega)) \), which yields (4.6). Moreover, the estimate (4.6) and (4.7) imply also that \( u^\varepsilon \) is bounded in \( L^q(0, T; W^1_0(\Omega)) \) \( \forall 1 < q < p - \frac{N}{N+1} \), according to the results in \([14,16,26,24]\).

Taking \( T_k(u^\varepsilon) \) as test function in (4.4) and we integrate in \([0,t]\), by assumptions (3.3), (3.5), (3.6), and by means of Young’s inequality one obtains
\[
\int_{\Omega} \frac{T_k(u^\varepsilon)}{2}(t) \, dx + \int_{0}^{t} \int_{\Omega} \frac{\partial b(x, u^\varepsilon)}{\partial s} | \nabla u^\varepsilon |^p \, dx \, dt \leq C \left( |G^\varepsilon|_{L^{p'}(Q)}^p + \| L \|_{L^{p'}(Q)}^p + \| \nabla g^\varepsilon \|_{L^p(Q)}^p + \| B \|_{L^p(Q)}^p \right) + k \| f \|_{L^1(\Omega)} + \| b(x, u_0^\varepsilon) \|_{L^1(\Omega)} + |\lambda^\varepsilon_{\nu} - \lambda^\varepsilon_{\nu} \|_{L^1(\Omega)}).
\]

where \( T_k(s) = \int_{0}^{s} \text{meas}(r) \, dr \, \forall s \in \mathbb{R} \). So that (4.9) and (4.10) hold true.

Now, by (4.38) and since \( \lambda^\varepsilon_{\nu}, \lambda^\varepsilon_{\nu} \) are bounded in \( L^1(Q) \), one obtains that \( \frac{\partial b(x, u^\varepsilon)}{\partial s} \) is bounded in \( L^1(0, T; W^{-1,1}(\Omega)) \). Moreover, by assumptions (3.2) and (3.3) we have that \( b(x, u^\varepsilon) \) is bounded in \( L^q(0, T; W^1_0(\Omega)) \) for every \( 1 < q < p - \frac{N}{N+1} \), so that using compactness arguments (see [42]) yield (4.11), and (4.12). Using (4.7) and (4.11) yield (4.13), while (4.14) and (4.15) derives from (4.10) and (4.12). Finally (3.1), (3.3), (4.11) and Lebesgue’s convergence theorem give (4.16).

Let us introduce for \( k \geq 0 \) fixed, the time regularization of the function \( T_k(u) \) in order to perform the monotonicity method. This kind of regularization has been first introduced by R. Landes. More recently, it has been exploited to solve a few nonlinear evolution problems with \( L^1 \) or measure data. This specific time regularization of \( T_k(u) \) (for fixed \( k \geq 0 \)) is defined as follows. Let \( (v'^\nu_0)_{\nu} \) in \( L^\infty(\Omega) \cap \bigcap W^{1,p}_0(\Omega) \) such that \( \| v'^\nu_0 \|_{L^\infty(\Omega)} \leq k \), for all \( \nu > 0 \), and \( v'^\nu_0 \to T_k(u_0) \) a.e. in \( \Omega \) with \( \frac{1}{\nu} \| v'^\nu_0 \|_{L^p(\Omega)} \to 0 \) as \( \nu \to +\infty \).

For fixed \( k \geq 0 \) and \( \nu > 0 \), let us consider the unique solution \( T_k(u)_{\nu} \in L^\infty(Q) \cap L^p(0,T, W^1_0(\Omega)) \) of the monotone problem:
\[
\frac{\partial T_k(u)_{\nu}}{\partial t} + \nu(T_k(u)_{\nu} - T_k(u)) = 0 \quad \text{in } D'(Q),
\]
\[
T_k(u)_{\nu}(t = 0) = v'^\nu_0 \quad \text{in } \Omega.
\]
The behavior of \( T_k(u)_{\nu} \) as \( \nu \to +\infty \) is investigated in [29] (see also [22]) and we just recall here that:
\[
T_k(u)_{\nu} \to T_k(u) \quad \text{strongly in } L^p(0,T, W^1_0(\Omega)) \quad \text{a.e. in } Q \quad \text{as } \nu \to +\infty
\]
with \( \| T_k(u)_{\nu} \|_{L^\infty(\Omega)} \leq k \) for any \( \nu > 0 \), and \( \frac{\partial T_k(u)_{\nu}}{\partial t} \in L^p(0,T, W^1_0(\Omega)) \).

Here and in the rest of paper \( \omega(\varepsilon, n, \delta, \mu) \) will indicate any quantity that vanishes as the parameters go to their limit point with in the same order in which they appear, that is, for example
\[
\lim_{\nu \to \infty} \lim_{\delta \to 0} \lim_{n \to \infty} \lim_{\varepsilon \to 0} | \omega(\varepsilon, n, \delta, \nu) | = 0.
\]

Now we give the basic result about approximate capacitary potential.
\textbf{Lemma 4.2.} Let $\mu_\delta = \mu_\delta^+ - \mu_\delta^-$ be concentrated respectively, on two disjoint $E^+$ and $E^-$ and $K_\delta = E^+ \cap K_\delta^-$ such that

\[
\mu_\delta^+(E^+ \setminus K_\delta^+) \leq \delta, \quad \mu_\delta^-(E^+ \setminus K_\delta^-) \leq \delta.
\]  

(4.19)

and there exist $\psi^+_\delta, \psi^-_\delta \in C_0^1(Q)$, such that

\[
\psi^+_\delta \equiv 1 \text{ and } \psi^-_\delta \equiv 1 \text{ respectively on } K_\delta^+ \text{ and } K_\delta^-,
\]

(4.20)

\[
0 \leq \psi^+_\delta, \quad \psi^-_\delta \leq 1,
\]

(4.21)

\[
supp(\psi^+_\delta) \cap supp(\psi^-_\delta) = \emptyset.
\]

(4.22)

Moreover

\[
\|\psi^+_\delta\|_8 \leq \delta, \quad \|\psi^-_\delta\|_8 \leq \delta,
\]

(4.23)

and in particular, there exists a decomposition of $(\psi^+_\delta)_t$ and a decomposition of $(\psi^-_\delta)_t$ such that

\[
\|\psi^+_\delta(t)\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq \frac{\delta}{3}, \quad \|\psi^-_\delta(t)\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq \frac{\delta}{3},
\]

(4.24)

\[
\|\psi^+_\delta(t)\|_{L^1(Q)} \leq \frac{\delta}{3}, \quad \|\psi^-_\delta(t)\|_{L^1(Q)} \leq \frac{\delta}{3}.
\]

(4.25)

Both $\psi^+_\delta$ and $\psi^-_\delta$ converges to zero weakly-* in $L^\infty(Q)$, in $L^1(Q)$, and up to subsequences, almost everywhere as $\delta$ vanishes. Moreover, if $\lambda^+_\delta$ and $\lambda^-_\delta$ are as in (4.1) we have

\[
\int_Q \psi^-_\delta \, d\lambda^+_\delta = \omega(\epsilon, \delta), \quad \int_Q \psi^+_\delta \, d\mu^+_\delta \leq \delta,
\]

(4.26)

\[
\int_Q \psi^+_\delta \, d\lambda^-_\delta = \omega(\epsilon, \delta), \quad \int_Q \psi^+_\delta \, d\mu^-_\delta \leq \delta,
\]

(4.27)

\[
\int_Q (1 - \psi^+_\delta \psi^-_\delta) \, d\lambda^+_\delta = \omega(\epsilon, \delta, \eta), \quad \int_Q (1 - \psi^+_\delta \psi^-_\delta) \, d\mu^+_\delta \leq \delta + \eta,
\]

(4.28)

\[
\int_Q (1 - \psi^+_\delta \psi^-_\delta) \, d\lambda^-_\delta = \omega(\epsilon, \delta, \eta), \quad \int_Q (1 - \psi^+_\delta \psi^-_\delta) \, d\mu^-_\delta \leq \delta + \eta.
\]

(4.29)

\textbf{Proof.} See [33], Lemma 5. \hfill \Box

In what follows we will always refer to subsequences of both $\psi^+_\delta$ and $\psi^-_\delta$ that satisfy all the convergence results stated in \textbf{Lemma 4.2.}

Now we will prove the following theorem

\textbf{Theorem 4.3.} Let $v^\varepsilon$ and $v$ be as before. Then, for every $k > 0$

\[
T_k(v^\varepsilon) \rightarrow T_k(v) \text{ strongly in } L^p(0,T;W^{1,p}_0(\Omega)).
\]

\textbf{Proof.} Let us first introduce the following function that we will use in the proof of \textbf{Theorem 4.3.}

\[
H_n(s) = \begin{cases} 
1 & \text{if } |s| \leq n, \\
\frac{2n - s}{2n} & \text{if } n < s \leq 2n, \\
\frac{n}{2n + s} & \text{if } -2n < s \leq -n, \\
0 & \text{if } |s| > 2n.
\end{cases}
\]

Let also introduce another auxiliary function in terms of $H_n$ by $B_n(s) = 1 - H_n(s)$. Our aim is to prove the following asymptotic estimate:

\[
\lim_{\varepsilon \rightarrow 0} \int_Q a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \, dxdt \leq \int_Q a(x, t, \nabla u) \nabla T_k(v) \, dx \, dt.
\]

(4.30)
In order to prove (4.30), we shall follow several steps.

\* Step 1.
For every \(\delta, \eta > 0\), let \(\psi^+_{\delta}, \psi^-_{\delta}\), \(\psi^+_{\eta}\) and \(\psi^-_{\eta}\) as in lemma 4.2 and let \(E^+\) and \(E^-\) be the sets where, respectively, \(\mu^+_{\delta}\) and \(\mu^-_{\delta}\) are concentrated. Setting \(\Phi_{\delta,\eta} = \psi^+_{\delta} \psi^+_{\eta} + \psi^-_{\delta} \psi^-_{\eta}\), we can write

\[
\int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v_\nu)) H_n(v^\varepsilon) \, dx \, dt
\]

\[
= \int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v_\nu)) H_n(v^\varepsilon) \Phi_{\delta,\eta} \, dx \, dt
\]

\[
+ \int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v_\nu)) H_n(v^\varepsilon)(1 - \Phi_{\delta,\eta}) \, dx \, dt.
\]

Now, if \(n > k\), since \(a(x, t, \nabla u^\varepsilon \chi_{\{|v^\varepsilon| \leq 2n\}}) \nabla T_k(v_\nu)\) is weakly compact in \(L^1(Q)\) as \(\varepsilon\) goes to zero, \(H_n(v^\varepsilon)\) converges to \(H_n(v)\) \(\ast\)-weakly in \(L^\infty(Q)\), and almost everywhere in \(Q\), by proposition 2.1 we have

\[
\lim_{\varepsilon \to 0} \int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v_\nu)) H_n(v^\varepsilon) \Phi_{\delta,\eta} \, dx \, dt
\]

\[
= \lim_{\varepsilon \to 0} \left[ \int_Q a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \Phi_{\delta,\eta} \, dx \, dt \right] - \int_Q \sigma_2 n \nabla T_k(v_\nu) H_n(v) \Phi_{\delta,\eta} \, dx \, dt
\]

\[
= \lim_{\varepsilon \to 0} \left[ \int_Q a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \Phi_{\delta,\eta} \, dx \, dt \right] - \int_Q \sigma_2 n \nabla T_k(v) \Phi_{\delta,\eta} \, dx \, dt + \omega(\nu).
\]

Since \(\Phi_{\delta,\eta}\) converges to zero \(\ast\)-weakly in \(L^\infty(Q)\) as \(\delta\) goes to zero,

\[
\int_Q \sigma_2 n \nabla T_k(v) \Phi_{\delta,\eta} \, dx \, dt = \omega(\delta).
\]

Therefore, if we prove that

\[
\lim_{n \to 0} \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_Q a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \Phi_{\delta,\eta} \, dx \, dt \leq 0,
\]

then we can conclude from (4.32) that

\[
\lim_{n \to 0} \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v_\nu)) H_n(v^\varepsilon) \Phi_{\delta,\eta} \, dx \, dt \leq 0.
\]

\* Step 2. Near to \(E\).
Before proving (4.33), we first show the following result

**Lemma 4.4.** Let \(u^\varepsilon\) be a solution of (4.3)-(4.5). Let \(\eta\) be a positive real number, and let \(\varphi^\eta_+\) and \(\varphi^\eta_-\) be two non negative functions in \(C_c^\infty(Q)\) such that

\[
0 \leq \varphi^\eta_+ \leq 1, \quad 0 \leq \varphi^\eta_- \leq 1,
\]

and

\[
0 \leq \int_Q \varphi^\eta_- \, d\mu^+_{\eta} \leq \eta, \quad 0 \leq \int_Q \varphi^\eta_+ \, d\mu^-_{\eta} \leq \eta,
\]

we then have

\[
\frac{1}{n} \int_{\{-2n < u^\varepsilon \leq -n\}} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \varphi^\eta_+ \, dx \, dt = \omega(\varepsilon, n, \eta),
\]

\[
\frac{1}{n} \int_{\{n < u^\varepsilon < 2n\}} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \varphi^\eta_- \, dx \, dt = \omega(\varepsilon, n, \eta).
\]
Proof. Let us prove (4.37): let $\beta_n(s) = B_n(s^+)$, we can choose $\beta_n(v^\varepsilon)\varphi_n^\varepsilon$ as test function in (4.4) and rearranging conveniently all terms we have

$$
\frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \varphi_n^\varepsilon \, dx \, dt + \int_Q \beta_n(v^\varepsilon) \varphi_n^\varepsilon \, d\lambda^\varepsilon
$$

$$
= \frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} a(x, t, \nabla u^\varepsilon) \nabla g^\varepsilon \varphi_n^\varepsilon \, dx \, dt
$$

$$
- \frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} a(x, t, \nabla u^\varepsilon) \nabla b(x, u^\varepsilon) \varphi_n^\varepsilon \, dx \, dt
$$

$$
+ \int_Q \beta_n(v^\varepsilon) \frac{d\varphi_n^\varepsilon}{dt} \, dx \, dt - \int_Q a(x, t, \nabla u^\varepsilon) \nabla \varphi_n^\varepsilon \beta_n(v^\varepsilon) \, dx \, dt + \int_Q f^\varepsilon \beta_n(v^\varepsilon) \varphi_n^\varepsilon \, dx \, dt
$$

$$
- \int_0^T \langle \text{div}(G^\varepsilon), \beta_n(v^\varepsilon) \varphi_n^\varepsilon \rangle \, dt + \int_Q \beta_n(v^\varepsilon) \varphi_n^\varepsilon \, d\lambda^\varepsilon,
$$

where $\beta_n(s) = \int_0^s \beta_n(r) \, dr$. Using the fact that $\int_Q \beta_n(v^\varepsilon) \varphi_n^\varepsilon \, d\lambda^\varepsilon \geq 0$ and by assumptions (3.2), (3.3), (3.5), (3.6) and Young’s inequality we obtain

$$
\frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \varphi_- \, dx \, dt
$$

$$
\leq \frac{C}{n} \int_Q \left( |\nabla g^\varepsilon|^p + |L^\varepsilon|^p + |B|^p \right) \, dx \, dt
$$

$$
+ \int_Q \beta_n(v^\varepsilon) \frac{d\varphi_n^\varepsilon}{dt} \, dx \, dt - \int_Q a(x, t, \nabla u^\varepsilon) \nabla \varphi_n^\varepsilon \beta_n(v^\varepsilon) \, dx \, dt
$$

$$
+ \int_Q f^\varepsilon \beta_n(v^\varepsilon) \varphi_n^\varepsilon \, dx \, dt
$$

$$
- \int_0^T \langle \text{div}(G^\varepsilon), \beta_n(v^\varepsilon) \varphi_n^\varepsilon \rangle \, dt + \int_Q \beta_n(v^\varepsilon) \varphi_n^\varepsilon \, d\lambda^\varepsilon.
$$

By (3.6) and (4.8) we have $a(x, t, \nabla u^\varepsilon)$ converges weakly in $(L^q(Q))^N$ as $\varepsilon$ goes to 0 for every $q' < 1 + (N+1)(p-1)$, since $\varphi_n^\varepsilon$ belongs to $C^\infty(Q)$ and $\beta_n(v^\varepsilon)$ converges to $\beta_n(v)$ a.e. in $Q$ and *-weakly in $L^\infty(Q)$ as $\varepsilon$ goes to zero and $\beta_n(v)$ converges to 0 a.e. in $Q$ and *-weakly in $L^\infty(Q)$ as $n$ goes to $+\infty$, thanks to proposition 2.1, we obtain

$$
\int_Q a(x, t, \nabla u^\varepsilon) \nabla \varphi_n^\varepsilon \beta_n(v^\varepsilon) \, dx \, dt = \omega(\varepsilon, n).
$$

Since $\beta_n(v^\varepsilon)$ converges to $\beta_n(v)$ in $L^1(Q)$ as $\varepsilon$ goes to 0, and $\beta_n(v)$ converges to 0 in $L^1(Q)$ as $n$ goes to $+\infty$, we obtain

$$
\int_Q \beta_n(v^\varepsilon) \frac{d\varphi_n^\varepsilon}{dt} \, dx \, dt = \omega(\varepsilon, n).
$$

Moreover, the weak $L^1(Q)$ convergence of $f^\varepsilon$ to $f$ and thanks to proposition 2.1 we obtain

$$
\int_Q f^\varepsilon \beta_n(v^\varepsilon) \varphi_n^\varepsilon \, dx \, dt = \omega(\varepsilon, n).
$$

Due the strong convergence of $\text{div}(G^\varepsilon)$ to $\text{div}(G)$ in $L^p(0, T, W^{1,p}(\Omega))$ and the weak convergene in $L^p(0, T, W_0^{1,p}(\Omega))$ of $\beta_n(v^\varepsilon)$ to $\beta_n(v)$ and $\beta_n(v)$ to 0 strongly in $L^p(0, T, W_0^{1,p}(\Omega))$ (this facts is an easy consequence of the estimate on the truncates of $u^\varepsilon$ in Proposition 4.1), we obtain

$$
\int_0^T \langle \text{div}(G^\varepsilon), \beta_n(v^\varepsilon) \varphi_n^\varepsilon \rangle \, dx \, dt = \omega(\varepsilon, n).
$$
Finally, by (4.35) and since $\beta_n(v^\varepsilon)$ is non negative and bounded and $\varphi^\eta$ is continuous, we have
\[
\int_Q \beta_n(v^\varepsilon) \varphi^\eta \, d\lambda_+^\varepsilon \leq \int_Q \varphi^\eta \, d\mu^+_\varepsilon + \omega(\varepsilon) = \omega(\varepsilon, \eta).
\]
Putting together all these facts lead to (4.37), while (4.36) can be obtained in the same way choosing $\beta_n(s) = B_n(s^-)$ and $\beta_n(v^\varepsilon) \varphi^\delta$ as test function in (4.4).

Now let us check (4.33). For fixed $k > 0$, we choose $(k - T_k(v^\varepsilon))H_n(v^\varepsilon)\psi^+_\delta \psi^+_\eta$ as test function in (4.4), defining $\Gamma_{n,k}(s) = \int_0^s (k - T_k(r))H_n(r) \, dr$, and integrating by parts, we obtain
\[
- \int_Q \Gamma_{n,k}(v^\varepsilon) \frac{d}{dt}(\psi^+_\delta \psi^+_\eta) \, dx dt \tag{4.38}
\]
\[
+ \int_Q (k - T_k(v^\varepsilon))H_n(v^\varepsilon) a(x,t,\nabla u^\varepsilon) \nabla (\psi^+_\delta \psi^+_\eta) \, dx dt
\]
\[
+ \int_Q a(x,t,\nabla u^\varepsilon) \nabla H_n(v^\varepsilon)(k - T_k(v^\varepsilon)) \psi^+_\delta \psi^+_\eta \, dx dt
\]
\[
- \int_Q a(x,t,\nabla u^\varepsilon) \nabla T_k(v^\varepsilon)H_n(v^\varepsilon) \psi^+_\delta \psi^+_\eta \, dx dt
\]
\[
= \int_Q f^\varepsilon H_n(v^\varepsilon)(k - T_k(v^\varepsilon)) \psi^+_\delta \psi^+_\eta \, dx dt - \int_0^T \langle \text{div}(G^\varepsilon), H_n(v^\varepsilon)(k - T_k(v^\varepsilon)) \psi^+_\delta \psi^+_\eta \rangle \, dt
\]
\[
+ \int_Q H_n(v^\varepsilon)(k - T_k(v^\varepsilon)) \psi^+_\delta \psi^+_\eta \, d\lambda_+^\varepsilon - \int_Q H_n(v^\varepsilon)(k - T_k(v^\varepsilon)) \psi^+_\delta \psi^+_\eta \, d\lambda_+^\varepsilon.
\]
For $n > k$, we have
\[
H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon| \leq k\}} = a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon| \leq k\}} \text{ a.e. in } Q,
\]
then rearranging all terms of (4.38), we obtain
\[
\int_Q a(x,t,\nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \psi^+_\delta \psi^+_\eta \, dx dt + \int_Q H_n(v^\varepsilon)(k - T_k(v^\varepsilon)) \psi^+_\delta \psi^+_\eta \, d\lambda_+^\varepsilon
\]
\[
= - \int_Q \Gamma_{n,k}(v^\varepsilon) \frac{d}{dt}(\psi^+_\delta \psi^+_\eta) \, dx dt + \frac{2k}{n} \int_{\{|-2n < \nu \leq -n\}} a(x,t,\nabla u^\varepsilon) \nabla v^\varepsilon \psi^+_\delta \psi^+_\eta \, dx dt
\]
\[
+ \int_Q (k - T_k(v^\varepsilon))H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon) \nabla (\psi^+_\delta \psi^+_\eta) \, dx dt
\]
\[
- \int_Q f^\varepsilon(k - T_k(v^\varepsilon))H_n(v^\varepsilon) \psi^+_\delta \psi^+_\eta \, dx dt
\]
\[
- \int_0^T \langle \text{div}(G^\varepsilon), H_n(v^\varepsilon)(k - T_k(v^\varepsilon)) \psi^+_\delta \psi^+_\eta \rangle \, dt + \int_Q (k - T_k(v^\varepsilon))H_n(v^\varepsilon) \psi^+_\delta \psi^+_\eta \, d\lambda_+^\varepsilon. \tag{4.39}
\]
Let us analyze term by term the right hand side of (4.39). Due to Proposition 4.1 we have $\Gamma_{n,k}(v^\varepsilon)$ converges to $\Gamma_{n,k}(v)$ weakly in $L^p(0,T;W^{1,p}_0(\Omega))$, and since $\Gamma_{n,k}(v) \in L^p(0,T;W^{1,p}_0(\Omega)) \cap L^\infty(Q)$, we deduce
\[
\int_Q \Gamma_{n,k}(v^\varepsilon) \frac{d}{dt}(\psi^+_\delta \psi^+_\eta) \, dx dt
\]
\[
= \int_Q \Gamma_{n,k}(v) \frac{d\psi^+_\delta \psi^+_\eta}{dt} \, dx dt + \int_Q \Gamma_{n,k}(v) \frac{d\psi^+_\delta \psi^+_\eta}{dt} \, dx dt + \omega(\varepsilon) = \omega(\varepsilon, \delta).
Since \((k - T_k(v^e))H_n(v^e)\) converges to \((k - T_k(v))H_n(v)\) a.e. and \(*-\) weakly in \(L^\infty(Q)\), thanks to Proposition 2.1, Proposition 4.1 and Lemma 4.2, we deduce
\[
\int_Q (k - T_k(v^e))H_n(v^e)a(x, t, \nabla u^e)\nabla(\psi_\delta^+ \psi_\eta^+) \, dx \, dt
\]
\[
= \int_Q (k - T_k(v))H_n(v)\sigma_{2n}\nabla(\psi_\delta^+ \psi_\eta^+) \, dx \, dt + \omega(\varepsilon) = \omega(\varepsilon, \delta).
\]
Moreover, \((k - T_k(v^e))H_n(v^e)\psi_\delta^+ \psi_\eta^+\) weakly converges to \((k - T_k(v))H_n(v)\psi_\delta^+ \psi_\eta^+\) in \(L^p(0, T; W_0^{1,p}(\Omega))\), and \(*-\) weakly in \(L^\infty(Q)\), thanks again to Lemma 4.2, we have
\[
\int_0^T (\text{div}(G^e), (k - T_k(v^e))H_n(v^e)\psi_\delta^+ \psi_\eta^+) \, dt = \omega(\varepsilon, \delta),
\]
and
\[
\int_Q f^e(k - T_k(v^e))H_n(v^e)\psi_\delta^+ \psi_\eta^+ \, dx \, dt = \omega(\varepsilon, \delta).
\]
Using assumptions (3.2), (3.3), (3.6), Young’s inequality, and since \(0 \leq \psi_\delta^+ \leq 1\) we obtain
\[
\frac{1}{n} \int_{\{-2n < v^e \leq -n\}} a(x, t, \nabla u^e)\nabla v^e \psi_\delta^+ \psi_\eta^+ \, dx \, dt \leq \frac{1}{n} \int_{\{-2n < v^e \leq -n\}} \frac{\partial b(x, u^e)}{\partial s}a(x, t, \nabla u^e)\nabla v^e \psi_\delta^+ \psi_\eta^+ \, dx \, dt + C n \int_Q \left( |\nabla g^e|^p + |L|^p + |B|^p \right) \, dx \, dt,
\]
applying Lemma 4.4 for \(\varphi_\eta^+ = \psi_\eta^+\), we obtain
\[
\frac{1}{n} \int_{\{-2n < v^e \leq -n\}} a(x, t, \nabla u^e)\nabla v^e \psi_\eta^+ \, dx \, dt = \omega(\varepsilon, n, \eta).
\]
Using (4.27) in Lemma 4.2, we have
\[
\left| \int_Q H_n(v^e)(k - T_k(v^e))\psi_\delta^+ \psi_\eta^+ \, d\lambda^e \right| \leq 2k \int_Q \psi_\delta^+ \psi_\eta^+ \, d\lambda^e
\]
\[
= 2k \int_Q \psi_\delta^+ \psi_\eta^+ \, d\mu^+ + \omega(\varepsilon) = \omega(\varepsilon, \delta).
\]
Collecting all we have shown above, we get
\[
\int_Q H_n(v^e)(k - T_k(v^e))\psi_\delta^+ \psi_\eta^+ \, d\lambda^e + \int_Q a(x, t, \nabla u^e)\nabla T_k(v^e)\psi_\delta^+ \psi_\eta^+ \, dx \, dt = \omega(\varepsilon, \delta, n, \eta).
\]
Since \(\int_Q H_n(v^e)(k - T_k(v^e))\psi_\delta^+ \psi_\eta^+ \, d\lambda^e \geq 0\), we obtain
\[
\int_Q a(x, t, \nabla u^e)\nabla T_k(v^e)\psi_\delta^+ \psi_\eta^+ \, dx \, dt \leq \omega(\varepsilon, \delta, \eta).
\]
On the other hand, reasoning as before with \((k + T_k(v^e))H_n(v^e)\psi_\delta^- \psi_\eta^-\) as test function we can obtain
\[
\int_Q a(x, t, \nabla u^e)\nabla T_k(v^e)\psi_\delta^- \psi_\eta^- \, dx \, dt \leq \omega(\varepsilon, \delta, \eta).
\]
Therefore, we obtain (4.33) which yields (4.34).
Remark 4.5. As we have shown above we have
\[
\int_Q H_n(\psi^\varepsilon)(k - T_k(\psi^\varepsilon))\psi_\delta^+\psi_\eta^+ d\lambda_+^\varepsilon + \int_Q \frac{\partial b(x, u^\varepsilon)}{\partial s}a(x, t, \nabla u^\varepsilon)\chi_{\{|v|\leq k\}}\nabla u^\varepsilon\psi_\delta^+\psi_\eta^+ dxdt
\]
\[
+ \int_Q a(x, t, \nabla u^\varepsilon)\chi_{\{|v|\leq k\}}\nabla b(x, u^\varepsilon)\psi_\delta^+\psi_\eta^+ dxdt
\]
\[
- \int_Q a(x, t, \nabla u^\varepsilon)\chi_{\{|v|\leq k\}}\nabla g^\varepsilon\psi_\delta^+\psi_\eta^+ dxdt = \omega(\varepsilon, \delta, n, \eta),
\]

by assumptions (3.2), (3.5), thanks to proposition 4.1 and Lemma 4.2 one obtains
\[
\int_Q H_n(\psi^\varepsilon)(k - T_k(\psi^\varepsilon))\psi_\delta^+\psi_\eta^+ d\lambda_+^\varepsilon = \omega(\varepsilon, \delta, n, \eta).
\]

Analogously we obtain
\[
\int_Q H_n(\psi^\varepsilon)(k + T_k(\psi^\varepsilon))\psi_\delta^-\psi_\eta^- d\lambda_-^\varepsilon = \omega(\varepsilon, \delta, n, \eta).
\]

The two last results above show an interesting property of approximating renormalized, they express the fact that $\psi^\varepsilon$ (and so the solution $u^\varepsilon$) is very large (greater than any $k > 0$) on the set where the singular measure $\mu^+_x$ is concentrated, and small (smaller than any $k < 0$) on the set where the singular measure $\mu^-_x$ is concentrated.

* Step 3. Far from $E$.
We first prove a result that will be essential to deal with the second term on the right hand side of (4.31).

Lemma 4.6. Let $k \geq 0$ be fixed. Let $S$ be an increasing $C^\infty(\mathbb{R})$-function such that $S(r) = r$ for $|r| \leq k$ and supp $S^\prime$ is compact. Then
\[
\int_0^T \int_0^t \langle \frac{\partial S(u^\varepsilon)}{\partial t}, (T_k(\psi^\varepsilon) - T_k(\psi)\nu)(1 - \Phi_{\delta, \eta}) \rangle \ dsdt \geq \omega(\varepsilon, \nu).
\]

Proof. The proof of the above Lemma follows the arguments in [8], Lemma 1 and we just sketch the proof of it.
Let $k \geq 0$ be fixed. Since $S$ is increasing and $S(r) = r$ for $|r| \leq k$,
\[
T_k(S(\psi^\varepsilon)) = T_k(\psi^\varepsilon) \quad \text{and} \quad T_k(S(\psi)) = T_k(\psi) \quad \text{a.e. in } Q.
\]

As a consequence $T_k(S(\psi))_\nu = T_k(\psi)_\nu$ a.e. in $Q$, for any $\nu > 0$.
It follows that under the notation $z^\varepsilon = S(\psi^\varepsilon)$ and $z = S(\psi)$, and thanks to properties of $T_k(z)_\nu$ we have
\[
\int_0^T \int_0^t \langle \frac{\partial S(u^\varepsilon)}{\partial t}, (T_k(\psi^\varepsilon) - T_k(\psi)\nu)(1 - \Phi_{\delta, \eta}) \rangle \ dsdt
\]
\[
= \int_0^T \int_0^t \langle \frac{\partial z^\varepsilon}{\partial t}, (T_k(z^\varepsilon) - T_k(z)\nu)(1 - \Phi_{\delta, \eta}) \rangle \ dsdt
\]
\[
= \int_0^T \int_0^t \langle \frac{\partial (z^\varepsilon - T_k(z)_\nu)}{\partial t}, (z^\varepsilon - T_k(z)_\nu)(1 - \Phi_{\delta, \eta}) \rangle \ dsdt
\]
\[
- \int_0^T \int_0^t \langle \frac{\partial z^\varepsilon}{\partial t}, (z^\varepsilon - T_k(z^\varepsilon))(1 - \Phi_{\delta, \eta}) \rangle \ dsdt
\]
\[
+ \int_0^T \int_0^t \langle \frac{\partial T_k(z)}{\partial t}, (z^\varepsilon - T_k(z)_\nu)(1 - \Phi_{\delta, \eta}) \rangle \ dsdt,
\]
integrating by parts we have

\[
\int_0^T \int_0^t \left( \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta,\eta}) \right) \, ds \, dt
\]

\(= \frac{1}{2} \int_0^T \int_0^t (z^\varepsilon - T_k(z)_\nu)^2 \frac{d\Phi_{\delta,\eta}}{dt} \, dx \, ds \, dt - \frac{1}{2} \int_0^T \int_\Omega (z^\varepsilon - T_k(z)_\nu)^2 \frac{d\Phi_{\delta,\eta}}{dt} \, dx \, ds \, dt
\]

\[
+ \frac{1}{2} \int_0^T \int_\Omega (z^\varepsilon - T_k(z)_\nu)^2 \, dx \, ds - \frac{T}{2} \int_\Omega (z^\varepsilon - T_k(z)_\nu)^2 (t = 0) \, dx
\]

\[
- \frac{1}{2} \int_0^T \int_\Omega (z - T_k(z))^2 \, dx \, ds + \frac{T}{2} \int_\Omega (z - T_k(z))^2 (t = 0) \, dx
\]

\[
+ \int_0^T \int_0^t \int_\Omega \frac{\partial T_k(z)}{\partial t}(z - T_k(z)_\nu)(1 - \Phi_{\delta,\eta}) \, dx \, ds \, dt
\]

since \(\int_0^r (s - T_k(s)) \, ds = \frac{1}{2}(r - T_k(r))^2\).

Using the definition of \(z^\varepsilon\) and \(z\), the fact that \(S\) is bounded and \(v^\varepsilon\) converges to \(v\) a.e. on \(Q\), we have \(z^\varepsilon\) converges to \(z\) strongly in \(L^2(Q)\) and in \(L^\infty(Q)\) strongly, the strong convergence of \(b(x, u^0)\) to \(b(x, u_0)\) in \(L^1(\Omega)\) implies that \(z^\varepsilon(t = 0)\) converges to \(S(b(x, u_0))\) strongly in \(L^2(\Omega)\).

Passing to the limit as \(\varepsilon\) tends to zero in (4.41) leads to

\[
\int_0^T \int_0^t \left( \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta,\eta}) \right) \, ds \, dt
\]

\(= \frac{1}{2} \int_0^T \int_\Omega (z - T_k(z)_\nu)^2 \frac{d\Phi_{\delta,\eta}}{dt} \, dx \, ds \, dt - \frac{1}{2} \int_0^T \int_\Omega (z - T_k(z))^2 \frac{d\Phi_{\delta,\eta}}{dt} \, dx \, ds \, dt
\]

\[
+ \frac{1}{2} \int_0^T \int_\Omega (z - T_k(z)_\nu)^2 \, dx \, ds - \frac{T}{2} \int_\Omega (z - T_k(z)_\nu)^2 (t = 0) \, dx
\]

\[
- \frac{1}{2} \int_0^T \int_\Omega (z - T_k(z))^2 \, dx \, ds + \frac{T}{2} \int_\Omega (z - T_k(z))^2 (t = 0) \, dx
\]

\[
+ \int_0^T \int_0^t \int_\Omega \frac{\partial T_k(z)}{\partial t}(z - T_k(z)_\nu)(1 - \Phi_{\delta,\eta}) \, dx \, ds \, dt + \omega(\varepsilon),
\]

by rewriting the definition of \(T_k(u)_\nu\) in terms of \(T_k(z)\) we have

\[
\frac{\partial T_k(z)}{\partial t} + \nu(T_k(z)_\nu - T_k(z)) = 0 \text{ in } D'(Q),
\]

\[
T_k(z)_\nu(t = 0) = v^\nu_0 \text{ in } \Omega.
\]

By properties of \(T_k(z)_\nu\) we obtain that \(T_k(z)_\nu\) converges to \(T_k(z)\) strongly in \(L^2(Q)\) and \(T_k(z)_\nu(t = 0)\) converges to \(T_k(S(b(x, u_0)))\) strongly in \(L^2(\Omega)\) as \(\nu\) tends to \(\infty\). Passing to the limit-inf as \(\nu\) tends to \(\infty\) in (4.42) leads to

\[
\lim_{\nu \to \infty} \lim_{\varepsilon \to 0} \int_0^T \int_0^t \left( \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta,\eta}) \right) \, ds \, dt
\]

\[
= \nu \int_0^T \int_0^t \int_\Omega (T_k(z) - T_k(z)_\nu)(z - T_k(z)_\nu)(1 - \Phi_{\delta,\eta}) \, dx \, ds \, dt.
\]

Thanks to definition of \(T_k(z)_\nu\) we have

\[
\int_0^T \int_0^t \int_\Omega (T_k(z) - T_k(z)_\nu)(z - T_k(z)_\nu)(1 - \Phi_{\delta,\eta}) \, dx \, ds \, dt
\]
we have

\[ \int_{|z| \leq k} (z - T_k(z)_\nu)(z - T_k(z)_\nu)(1 - \Phi_{\delta, \eta}) \, dx \, ds \, dt \]

and the three terms are all non negatives, then

\[ \int_0^T \int_0^T \int_0^T (\frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta})) \, dt \geq \omega(\varepsilon, \nu) \]

Now, let us multiply by \( H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta}) \) the equation solved by \( u^\varepsilon \) and integrate to obtain

\[ \int_0^T \int_Q a(x,t,\nabla u^\varepsilon) \cdot \nabla (T_k(v^\varepsilon) - T_k(v)_\nu) H_n(v^\varepsilon)(1 - \Phi_{\delta, \eta}) \, dx \, dt \]

\[ = \int_0^T \langle \text{div}(G^\varepsilon), H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta}) \rangle \, dt \]

\[ + \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta}) \, d\lambda^\varepsilon_+ \]

\[ + \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta}) \, d\lambda^\varepsilon_- \]

Let us analyze term by term the identity (4.43), by Lemma 4.6 we have

\[ \int_0^T \int_Q \langle \frac{\partial u^\varepsilon}{\partial t}, H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta}) \rangle \, dt \geq \omega(\varepsilon, \nu). \]

The almost everywhere and \( * \)-weak convergence of \( H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu) \) to \( H_n(v)(T_k(v) - T_k(v)_\nu) \) in \( L^\infty(Q) \), the properties of \( T_k(v)_\nu \) and thanks to Propositions 2.1 and 4.1 we have

\[ \int_Q a(x,t,\nabla u^\varepsilon) \cdot \nabla \Phi_{\delta, \eta} H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu) \, dx \, dt = \omega(\varepsilon, \nu). \]

Due the strong convergence of \( \text{div}(G^\varepsilon) \) to \( \text{div}(G) \) in \( \L^p(0,T, W^{-1,p}(\Omega)) \), Proposition 4.1 and the properties of \( T_k(v)_\nu \) one obtains

\[ \int_0^T \langle \text{div}(G^\varepsilon), H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta}) \rangle \, dt = \omega(\varepsilon, \nu). \]
The weak convergence of $f^\varepsilon$ to $f$ in $L^1(Q)$, the almost everywhere and $\ast-$ weak convergence of $H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)$ to $H_n(v)(T_k(v) - T_k(v)_\nu)$ in $L^\infty(Q)$, Propositions 2.1, the properties of $T_k(v)_\nu$ and the Lebesgue’s dominated convergence theorem leads to

$$\int_Q f^\varepsilon H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta})
\, dx dt = \omega(\varepsilon, \nu).$$

By Lemma 4.2 and the fact that $|H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)| \leq 2k$ we obtain

$$\left| \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta}) \, d\lambda_+^\varepsilon \right|,$$

$$\leq 2k \int_Q (1 - \psi_+^\varepsilon \psi_-^\eta) \, d\lambda_+^\varepsilon + 2k \int_Q \psi_-^\varepsilon \psi_-^\eta \, d\lambda_+^\varepsilon,$$

and

$$\int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta}) \, d\lambda_+^\varepsilon = \omega(\varepsilon, \delta, \eta),$$

and similarly we get

$$\int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta}) \, d\lambda_-^\varepsilon = \omega(\varepsilon, \delta, \eta).$$

It remains to prove that

$$\int_Q a(x, t, \nabla u^\varepsilon) \nabla H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta}) \, dx dt = \omega(\varepsilon, n, \delta, \eta).$$

We have

$$\left| \frac{1}{n} \int_{\{n \leq |v^\varepsilon| < 2n\}} a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon(T_k(v^\varepsilon) - T_k(v)_\nu)(1 - \Phi_{\delta, \eta}) \, dx \, dt \right|,$$

$$\leq 2k \int_{\{n \leq |v^\varepsilon| < 2n\}} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon(1 - \Phi_{\delta, \eta}) \, dx \, dt,$$

$$+ \frac{C}{n} \int_Q \left( |\nabla g^\varepsilon|^2 + |L|^2 + |B|^2 \right) \, dx dt = I_1 + I_2,$$

we have $I_2 = \omega(n)$, and we rewrite $I_1$ as follows:

$$I_1 = \frac{2k}{n} \int_{\{n \leq v^\varepsilon < 2n\}} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon(1 - \psi_+^\delta \psi_-^\eta) \, dx \, dt,$$

$$- \frac{2k}{n} \int_{\{n \leq v^\varepsilon < 2n\}} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \psi_+^\delta \psi_-^\eta \, dx \, dt,$$

$$+ \frac{2k}{n} \int_{\{-2n < v^\varepsilon \leq -n\}} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon(1 - \psi_-^\delta \psi_-^\eta) \, dx \, dt,$$

$$- \frac{2k}{n} \int_{\{-2n < v^\varepsilon \leq -n\}} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \psi_+^\delta \psi_+^\eta \, dx \, dt.$$

We can apply Lemma 4.4 for every term above. Indeed, if we define $\varphi_{\delta, \eta}^\varepsilon = 1 - \psi_+^\delta \psi_-^\eta$, we have by Lemma 4.2,

$$\int_Q \varphi_{\delta, \eta}^\varepsilon \, d\mu_+^\delta \leq \eta + \delta,$$
then $\varphi_{-\eta,\eta}$ satisfies (4.33), thanks to Lemma 4.4 we obtain
\[
\frac{2k}{n} \int_{\{n \leq u^\varepsilon < 2n\}} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon (1 - \psi_{\varepsilon, \eta}^+) dxdt \leq \omega(\varepsilon, n) + \delta + \eta
\]
\[
= \omega(\varepsilon, n, \delta, \eta).
\]

In analogous way we obtain the same result for the others terms. Therefore, we obtain our estimate far from $E$
\[
\int_{Q} a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v)) H_n(v^\varepsilon)(1 - \Phi_{\delta, \eta}) dxdt \leq \omega(\varepsilon, \nu, n, \delta, \eta). \tag{4.44}
\]
\[\square\]

**Step 4.** Strong convergence of truncates.

Collecting together (4.31), (4.34) and (4.44), we have by taking again $n > k$,
\[
\lim_{\varepsilon \to 0} \int_{Q} a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) dxdt \leq \int_{Q} \sigma_k \nabla T_k(v) dxdt. \tag{4.45}
\]

Now, we prove that
\[
\lim_{\varepsilon \to 0} \int_{Q} \frac{\partial b(x, u^\varepsilon)}{\partial s} \left[ a(x, t, \nabla u^\varepsilon, \chi_{\{u^\varepsilon \leq k\}}) - a(x, t, \nabla u) \chi_{\{u \leq k\}} \right] dxdt
\]
\[
\times \left[ \nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}} - \nabla u \chi_{\{|u| \leq k\}} \right] dxdt = 0. \tag{4.46}
\]
We set
\[
A^\varepsilon = \int_{Q} \frac{\partial b(x, u^\varepsilon)}{\partial s} \left[ a(x, t, \nabla u^\varepsilon, \chi_{\{u^\varepsilon \leq k\}}) - a(x, t, \nabla u) \chi_{\{u \leq k\}} \right] dxdt
\]
\[
\times \left[ \nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}} - \nabla u \chi_{\{|u| \leq k\}} \right] dxdt.
\]
We split (4.38), into $A^\varepsilon = A_1 + A_2 + A_3$, where
\[
A_1^\varepsilon = \int_{Q} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}} dxdt,
\]
\[
A_2^\varepsilon = -\int_{Q} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u^\varepsilon) \nabla u \chi_{\{|u| \leq k\}} \chi_{\{|u^\varepsilon| \leq k\}} dxdt,
\]
\[
A_3^\varepsilon = -\int_{Q} \frac{\partial b(x, u^\varepsilon)}{\partial s} a(x, t, \nabla u) \left( \nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}} - \nabla u \chi_{\{|u| \leq k\}} \right) dxdt.
\]
We pass to the limit as $\varepsilon$ tends to 0 in $A_1, A_2$ and $A_3$. Let us remark that we have $\frac{\partial b(x, u^\varepsilon)}{\partial s} \nabla u^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}} = \nabla T_k(v^\varepsilon) - \nabla \chi_{\{|u^\varepsilon| \leq k\}} b(x, u^\varepsilon) \chi_{\{|u^\varepsilon| \leq k\}} + \nabla g^\varepsilon \chi_{\{|u^\varepsilon| \leq k\}}$ a.e in $Q$, and we have also $\chi_{\{|u^\varepsilon| \leq k\}}$ almost everywhere converges to $\chi_{\{|u| \leq k\}}$ in $Q$ (see [14]), we obtain:
\[
\lim_{\varepsilon \to 0} A_1^\varepsilon \tag{4.47}
\]
\[
= \lim_{\varepsilon \to 0} \int_{Q} a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) dxdt + \lim_{\varepsilon \to 0} \int_{Q} a(x, t, \nabla u^\varepsilon) \chi_{\{|u^\varepsilon| \leq k\}} \nabla g^\varepsilon dxdt
\]
\[
\leq \int_{Q} \sigma_k \nabla T_k(v) dxdt - \int_{Q} \sigma_k \nabla \chi_{\{|u| \leq k\}} b(x, u) \chi_{\{|u| \leq k\}} dxdt + \int_{Q} \sigma_k \nabla \chi_{\{|u| \leq k\}} dxdt.
\]
As a consequence of Proposition 4.1, we deduce that

\[
\lim_{\varepsilon \to 0} A_2^\varepsilon = - \int_Q \sigma_k (\nabla T_k (v) - \nabla b (x, u) + \nabla g) \, dx \, dt,
\]

(4.48)

and

\[
\lim_{\varepsilon \to 0} A_3^\varepsilon = - \lim_{\varepsilon \to 0} \int_Q a (x, t, \nabla u) \left( \nabla T_k (v^\varepsilon) - (\nabla b (x, u^\varepsilon) + \nabla g^\varepsilon) \chi_{\{|v^\varepsilon| \leq k\}} \right)
- \frac{\partial b (x, u^\varepsilon)}{\partial s} \left( \nabla T_k (v^\varepsilon) - (\nabla b (x, u^\varepsilon) + \nabla g^\varepsilon) \chi_{\{|v^\varepsilon| \leq k\}} \right) \, dx \, dt = 0.
\]

(4.49)

Therefore collecting (4.47), (4.48) and (4.49) yield (4.46). Using (4.45), (4.46) and the usual Minty’s argument we deduce that, \( \sigma_k = a (x, t, \nabla u) \chi_{\{|v| \leq k\}} \). Through the monotonicity argument which relies on (3.7) (see [17], Lemma 5), we can deduce from (4.46) that

\[
\nabla u^\varepsilon \chi_{\{|v^\varepsilon| \leq k\}} \to \nabla u \chi_{\{|v| \leq k\}} \text{ a.e. in } Q,
\]

and since \( a (x, t, \nabla u^\varepsilon) \nabla u^\varepsilon \chi_{\{|v^\varepsilon| \leq k\}} \) converges to \( a (x, t, \nabla u) \nabla u \chi_{\{|v| \leq k\}} \) weakly in \( L^1 (Q) \), by coercivity we have that \( |\nabla u^\varepsilon| \chi_{\{|v^\varepsilon| \leq k\}} \) is equi-integrable, as a consequence of Vitali’s theorem and since \( g^\varepsilon \) strongly converges in \( L^p (0, T; W_0^{1,p} (\Omega)) \) yields

\[
T_k (v^\varepsilon) \to T_k (v) \text{ strongly in } L^p (0, T; W_0^{1,p} (\Omega)),
\]

the proof of Theorem 4.3, is complete. \( \square \)

Proof. (Proof of Theorem 4.1). Now we are able to prove that Problem (1.1)-(1.3) has a renormalized solutions.

Let \( S \) in \( W^{2, \infty} (\mathbb{R}) \), such that \( S' \) has a compact support as in Definition 3.1, and let \( \varphi \in C_c^\infty (Q) \), then the approximating solutions \( u^\varepsilon \) (and \( v^\varepsilon \)) satisfy

\[
- \int_0^T \langle \varphi_\varepsilon, S (v^\varepsilon) \rangle \, dt + \int_Q S' (v^\varepsilon) a (x, t, \nabla u^\varepsilon) \nabla \varphi \, dx \, dt + \int_Q S'' (v^\varepsilon) a (x, t, \nabla u^\varepsilon) \nabla v^\varepsilon \varphi \, dx \, dt
= \int_Q f^\varepsilon S' (v^\varepsilon) \varphi \, dx \, dt + \int_Q G^\varepsilon S' (v^\varepsilon) \nabla \varphi \, dx \, dt + \int_Q S'' (v^\varepsilon) G^\varepsilon \nabla v^\varepsilon \varphi \, dx \, dt
+ \int_Q S' (v^\varepsilon) \varphi \, d\lambda^\varepsilon_+ - \int_Q S' (v^\varepsilon) \varphi \, d\lambda^\varepsilon_-
\]

(4.50)

Thanks to Theorem 4.3, all terms in (4.50) easily pass to the limit on \( \varepsilon \) except the last two terms that give some problem. We can write following the arguments in [33] we have

\[
\int_Q S' (v^\varepsilon) \varphi \, d\lambda^\varepsilon_+ = \int_Q S' (v^\varepsilon) \varphi \psi^\varepsilon_+ \, d\lambda^\varepsilon_+ + \int_Q S' (v^\varepsilon) \varphi (1 - \psi^\varepsilon_+) \, d\lambda^\varepsilon_+.
\]

(4.51)

Let \( \psi^\varepsilon_+ \) be defined as in Lemma 4.2, then we have

\[
\int_Q S' (v^\varepsilon) \varphi (1 - \psi^\varepsilon_+) \, d\lambda^\varepsilon_+ \leq C \int_Q (1 - \psi^\varepsilon_+) \, d\lambda^\varepsilon_+ = \omega (\varepsilon, \delta),
\]

while choosing \( S' (v^\varepsilon) \varphi \psi^\varepsilon_+ \) in (4.4) one gets,

\[
\int_Q S' (v^\varepsilon) \varphi \psi^\varepsilon_+ \, d\lambda^\varepsilon_+ = - \int_Q f^\varepsilon S' (v^\varepsilon) \varphi \psi^\varepsilon_+ \, dx \, dt - \int_Q G^\varepsilon S' (v^\varepsilon) \nabla (\varphi \psi^\varepsilon_+) \, dx \, dt
- \int_Q G^\varepsilon S'' (v^\varepsilon) \nabla \varphi \psi^\varepsilon_+ \, dx \, dt + \int_Q S' (v^\varepsilon) \varphi \psi^\varepsilon_+ \, d\lambda^\varepsilon_- - \int_Q S (v^\varepsilon) (\varphi \psi^\varepsilon_+) \, d\lambda^\varepsilon_-
\]

(4.52)
Now, thanks to Proposition 4.1 and the properties of $\psi_\delta^+$, we have

$$\int_Q f^\varepsilon S'(\varepsilon) \varphi \psi_\delta^+ \, dxdt = \omega(\varepsilon, \delta)$$
and

$$\int_Q G^\varepsilon S'(\varepsilon) \nabla(\varphi \psi_\delta^+) \, dxdt = \omega(\varepsilon, \delta).$$

By Lemma 4.2, we deduce

$$\left| \int_Q S'(\varepsilon) \varphi \psi_\delta^+ \, d\lambda^+ \right| \leq C \int_Q \psi_\delta^+ \, d\lambda^- = \omega(\varepsilon, \delta).$$

Again by Lemma 4.2, and since $S(v) \in L^p(0, T; W^{1,p}_0(\Omega)) \cap L^\infty(Q)$,

$$\int_Q S(v)\varphi \psi_\delta^+ \, dxdt = \omega(\varepsilon, \delta).$$

By Theorem 4.3 and Lemma 4.2, we have

$$\int_Q S'(\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla(\psi_\delta^+ \varphi) \, dxdt = \omega(\varepsilon, \delta),$$

and

$$\int_Q S''(\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon \psi_\delta^+ \varphi \, dxdt = \omega(\varepsilon, \delta).$$

Therefore, from (4.52) we deduce

$$\int_Q S'(\varepsilon) \varphi \, d\lambda^+ = \omega(\varepsilon).$$

Similarly, we can prove that

$$\int_Q S'(\varepsilon) \varphi \, d\lambda^- = \omega(\varepsilon).$$

As a consequence of the above convergence results, we are in a position to pass to the limit as $\varepsilon$ tends to 0 in (4.50) and to conclude that $u$ satisfies (3.11).

It remains to show that $S(v)$ satisfies the initial condition (3.12). To this end, firstly remark that $S'(v)$ being bounded in $L^\infty(Q)$, secondly, (4.50) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial S(v)}{\partial t}$ is bounded in $L^1(Q) + L^p(0, T; W^{-1,p}(\Omega))$.

As a consequence, an Aubin’s type lemma (see e.g. [42], Corollary 4) implies that $S(v)$ lies in a compact set of $C([0, T]; W^{-1,s}(\Omega))$ for any $s < \inf(p', \frac{1}{N-1})$. It follows that, on one hand, $S(v)(t = 0)$ converges to $S(v)(t = 0)$ strongly in $W^{-1,s}(\Omega)$, On the other hand, the smoothness of S imply that $S(v)(t = 0)$ converges to $S(b(x,u))(t = 0)$ strongly in $L^q(\Omega)$, and this is (3.12). Due to (4.2), we conclude that $S(v)(t = 0) = S(b(x,u_0))$ converges to $S(b(x,u_0)(t = 0)$ strongly in $L^q(\Omega)$. Then $v$ satisfies (3.12).

Now choosing $\beta_n(v)$ as test function in (4.4) where $\varphi \in C_c^\infty(Q)$, we obtain

$$-\int_0^T \langle \varphi_t, \beta_n(v) \rangle \, dt + \int_Q \beta_n(v) a(x, t, \nabla u^\varepsilon) \nabla \varphi \, dxdt$$

$$+ \frac{1}{n} \int_{\{\varepsilon \leq v^\varepsilon < 2n\}} a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon \varphi \, dxdt$$

$$= \int_Q f^\varepsilon \beta_n(v) \varphi \, dxdt - \int_0^T \langle \text{div}(G^\varepsilon), \beta_n(v^\varepsilon) \varphi \rangle \, dxdt$$

$$+ \int_Q \beta_n(v^\varepsilon) \varphi \, d\lambda^+ - \int_Q \beta_n(v^\varepsilon) \varphi \, d\lambda^-.$$
Reasoning as before (in particular as in the proof of Lemma 4.4) we obtain

\[
\int_0^T \langle \varphi_t, \beta_n(\varphi^\varepsilon) \rangle \, dt = \omega(\varepsilon, n), \quad \int_Q \beta_n(\varphi^\varepsilon) a(x, t, \nabla u^\varepsilon) \cdot \nabla \varphi \, dxdt = \omega(\varepsilon, n),
\]

\[
\int_Q f^\varepsilon \beta_n(\varphi^\varepsilon) \varphi \, dxdt = \omega(\varepsilon, n), \quad \int_0^T \langle \text{div}(G^\varepsilon), \beta_n(\varphi^\varepsilon) \varphi \rangle \, dxdt = \omega(\varepsilon, n).
\]

Thanks to Theorem 4.3 we have

\[
\frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon \varphi \, dxdt = \frac{1}{n} \int_{\{n \leq v \leq 2n\}} a(x, t, \nabla u) \nabla v \varphi \, dxdt + \omega(\varepsilon).
\]

Now we deal with the two last terms in the right hand side of (4.55) we can write

\[
\int_Q \beta_n(\varphi^\varepsilon) \varphi \, d\lambda^\delta_+ = -\int_Q h_n(\varphi^\varepsilon) \varphi \, d\lambda^\delta_+ + \int_Q \varphi \, d\lambda^\delta_+
\]

where \(h_n(s) = H_n(s^+)\). By construction of \(\lambda^\delta_+\) we have

\[
\int_Q \varphi \, d\lambda^\delta_+ = \int_Q \varphi \, d\mu^\delta_+ + \omega(\varepsilon).
\]

Following the same argument as in (4.50) and (4.51) by taking \(h_n(\varphi^\varepsilon) = S^\epsilon(\varphi^\varepsilon)\) we obtain

\[
\int_Q h_n(\varphi^\varepsilon) \varphi \, d\lambda^\delta_+ = \omega(\varepsilon).
\]

If we prove that

\[
\int_Q \beta_n(\varphi^\varepsilon) \varphi \, d\lambda^\delta_- = \omega(\varepsilon), \tag{4.56}
\]

then, we obtain for every \(\varphi \in C^\infty_\varepsilon(Q)\)

\[
\lim_{n \to \infty} \frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla v \varphi \, dxdt = \int_Q \varphi \, d\mu^\delta_+. \tag{4.57}
\]

We can write

\[
\int_Q \beta_n(\varphi^\varepsilon) \varphi \, d\lambda^\delta_- = \int_Q \beta_n(\varphi^\varepsilon) \varphi \psi^\delta_- \, d\lambda^\delta_- + \int_Q \beta_n(\varphi^\varepsilon) \varphi(1 - \psi^\delta_-) \, d\lambda^\delta_-,
\]

by Lemma 4.2, we obtain

\[
\int_Q \beta_n(\varphi^\varepsilon) \varphi(1 - \psi^\delta_-) \, d\lambda^\delta_- = \omega(\varepsilon, \delta).
\]

Choosing \(\beta_n(\varphi^\varepsilon) \varphi \psi^\delta_-\) as a test function in the formulation of \(u^\varepsilon\)

\[
\int_Q \beta_n(\varphi^\varepsilon) \varphi \psi^\delta_- \, d\lambda^\delta_- = \int_0^T \langle (\varphi \psi^\delta_-)_t, \overline{\beta_n(\varphi)} \rangle \, dt - \int_Q \beta_n(\varphi^\varepsilon) a(x, t, \nabla u^\varepsilon) \nabla (\varphi \psi^\delta_-) \, dxdt
\]

\[
- \frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon \varphi \psi^\delta_- \, dxdt + \int_Q f^\varepsilon \beta_n(\varphi^\varepsilon) \varphi \psi^\delta_- \, dxdt
\]

\[
+ \int_Q G^\varepsilon \beta_n(\varphi^\varepsilon) \nabla (\varphi \psi^\delta_-) \, dxdt + \frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} G^\varepsilon \nabla v^\varepsilon \varphi \psi^\delta_- \, dxdt + \int_Q \beta_n(\varphi^\varepsilon) \varphi \psi^\delta_- \, d\lambda^\delta_+.
\]

Using again Proposition 2.1, Proposition 4.1, Lemma 4.2 and Lemma 4.4 yields (4.56), and therefore we obtain (4.57) for every \(\varphi \in C^\infty_\varepsilon(Q)\). Now if \(\varphi \in C^\infty(Q)\), we can split

\[
\frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla v \varphi \, dxdt = \frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla v \varphi \psi^\delta_+ \, dxdt \tag{4.58}
\]
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\[ + \frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla v \varphi(1 - \psi^+_\delta) \, dx \, dt \, . \]

Thanks to (4.57), we have

\[ \lim_{n \to \infty} \frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla v \varphi \psi^+_\delta \, dx \, dt = \int_Q \varphi \, d\mu^+_\varepsilon + \omega(\delta), \]

By Lemma 4.4, we obtain

\[ \frac{1}{n} \int_{\{n \leq v^\varepsilon < 2n\}} a(x, t, \nabla u^\varepsilon) \nabla v^\varepsilon \varphi(1 - \psi^+_\delta) \, dx \, dt = \omega(\varepsilon, n, \delta). \]

Thanks to Theorem 4.3, we deduce

\[ \frac{1}{n} \int_{\{n \leq v < 2n\}} a(x, t, \nabla u) \nabla v \varphi(1 - \psi^+_\delta) \, dx \, dt = \omega(n, \delta). \]

Putting together all these facts above, we get (3.13) for every \( \varphi \in C^\infty(\overline{Q}) \), and by density argument (3.13) holds for every \( \varphi \in C(\overline{Q}) \). To obtain (3.14) we can reason as before using \( \psi^+_\delta \) in the place of \( \psi^-_\delta \) and viceversa, and this conclude the proof of Theorem 4.1.

\[ \square \]

References


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