



On Pencil of Bounded Linear Operators on Non-archimedean Banach Spaces

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ABSTRACT: In this paper, we introduce and check some properties of pseudospectrum and some approximation of a pencil of bounded linear operators on non-archimedean Banach spaces. Our main result extend some results for a pencil of bounded linear operators on non-archimedean Banach spaces and we give some examples to support our work.

Key Words: Non-archimedean Banach spaces, spectrum, pencil of linear operator, pseudospectrum.

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1. Introduction

Throughout this paper, X is a non-archimedean (n.a) Banach space over a (n.a) non trivially complete valued field \mathbb{K} with valuation $|\cdot|$, $\mathcal{L}(X)$ denotes the set of all bounded linear operators on X , \mathbb{Q}_p is the field of p -adic numbers ($p \geq 2$ being a prime) equipped with p -adic valuation $|\cdot|_p$, \mathbb{Z}_p denotes the ring of p -adic integers of \mathbb{Q}_p (is the unit ball of \mathbb{Q}_p). We denote the completion of algebraic closure of \mathbb{Q}_p under the p -adic valuation $|\cdot|_p$ by \mathbb{C}_p . For more details, we refer to [4] and [8].

Remember that a free Banach space X is a non-archimedean Banach space for which there exists a family $(e_i)_{i \in \mathbb{N}}$ in $X \setminus \{0\}$ such that every element $x \in X$ can be written in the form of a convergent sum $x = \sum_{i \in \mathbb{N}} x_i e_i$, $x_i \in \mathbb{K}$ and $\|x\| = \sup_{i \in \mathbb{N}} |x_i| \|e_i\|$. The family $(e_i)_{i \in \mathbb{N}}$ is called an orthogonal basis. In free

Banach space X , each bounded linear operator A on X can be written in a unique fashion as a pointwise convergent series, that is, there exists an infinite matrix $(a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ with coefficients in \mathbb{K} such that

$$A = \sum_{i,j \in \mathbb{N}} a_{i,j} e'_j \otimes e_i, \text{ and } \forall j \in \mathbb{N}, \lim_{i \rightarrow \infty} |a_{i,j}| \|e_i\| = 0,$$

where $(\forall j \in \mathbb{N}) e'_j(u) = u_j$ (e'_j is the linear form associated with e_j).

Moreover, for each $j \in \mathbb{N}$, $Ae_j = \sum_{i \in \mathbb{N}} a_{i,j} e_i$ and its norm is defined by

$$\|A\| = \sup_{i,j} \frac{|a_{i,j}| \|e_i\|}{\|e_j\|}.$$

Also, recall that X is of countable type if it contains a countable set whose linear hull is dense in X . For more details, we refer [4] and [8]. An unbounded linear operator $A : D(A) \subseteq X \rightarrow Y$ is said to be closed if for all $(x_n)_{n \in \mathbb{N}} \subset D(A)$ such that $\|x_n - x\| \rightarrow 0$ and $\|Ax_n - y\| \rightarrow 0$ as $n \rightarrow \infty$, for some $x \in X$ and $y \in Y$, then $x \in D(A)$ and $y = Ax$. The collection of closed linear operators from X into Y is denoted by $\mathcal{C}(X, Y)$. When $X = Y$, $\mathcal{C}(X, X) = \mathcal{C}(X)$. If $A \in \mathcal{L}(X)$ and B is an unbounded linear operator, then $A + B$ is closed if and only if B is closed, for more details, we refer to [4].

Definition 1.1. [4] Let $\omega = (\omega_i)_i$ be a sequence of non-zero elements of \mathbb{K} . We define \mathbb{E}_ω by

$$\mathbb{E}_\omega = \{x = (x_i)_i : \forall i \in \mathbb{N}, x_i \in \mathbb{K}, \text{ and } \lim_{i \rightarrow \infty} |\omega_i|^{\frac{1}{2}} |x_i| = 0\},$$

and it is equipped with the norm

$$\forall x \in \mathbb{E}_\omega : x = (x_i)_i, \|x\| = \sup_{i \in \mathbb{N}} (|\omega_i|^{\frac{1}{2}} |x_i|).$$

Remark 1.2. [4]

(i) The space $(\mathbb{E}_\omega, \|\cdot\|)$ is a non-archimedean Banach space.

(ii) If

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega &\longrightarrow \mathbb{K} \\ (x, y) &\longmapsto \sum_{i=0}^{\infty} x_i y_i \omega_i, \end{aligned}$$

where $x = (x_i)_i$ and $y = (y_i)_i$. Then, the space $(\mathbb{E}_\omega, \|\cdot\|, \langle \cdot, \cdot \rangle)$ is called a *p-adic* (or non-archimedean) Hilbert space.

(iii) The orthogonal basis $\{e_i, i \in \mathbb{N}\}$ is called the canonical basis of \mathbb{E}_ω , where for all $i \in \mathbb{N}$, $\|e_i\| = |\omega_i|^{\frac{1}{2}}$.

Definition 1.3. [7] Let X, Y, Z be three non-archimedean Banach spaces over \mathbb{K} , let $A \in B(X, Y)$ and $B \in B(X, Z)$. Then A majorizes B , if there exists $M > 0$ such that

$$\text{for all } x \in X, \|Bx\| \leq M \|Ax\|. \quad (1.1)$$

Theorem 1.4. [7] Assume that, either field \mathbb{K} is spherically complete or both Y and Z are countable type Banach spaces over \mathbb{K} . Let $A \in B(X, Y)$ and $B \in B(X, Z)$. Then the statements are equivalent:

- (1) $R(A^*) \subset R(B^*)$;
- (2) B majorizes A ;
- (3) there exists a continuous linear operator $D : R(B) \longrightarrow Y$, such that $A = DB$.

In this paper, we study the problem of finding the eigenvalues of the generalized eigenvalue problem

$$Ax = \lambda Bx$$

for $\lambda \in \mathbb{K}$, $x \in X$, and $A, B \in \mathcal{L}(X)$.

For more basic concepts of non-archimedean operators theory, we refer to [4]. In [2], the authors extended the notion of pseudospectrum of linear operator A on non-archimedean Banach space X as follows.

Definition 1.5. [2] Let X be a non-archimedean Banach space over \mathbb{K} and $\varepsilon > 0$. The pseudospectrum of a linear operator A on X is defined by

$$\sigma_\varepsilon(A) = \sigma(A) \cup \{\lambda \in \mathbb{K} : \|(\lambda - A)^{-1}\| > \varepsilon^{-1}\},$$

by convention $\|(\lambda - A)^{-1}\| = \infty$ if, and only if, $\lambda \in \sigma(A)$.

2. Main results

We introduce the following definitions.

Definition 2.1. Let X be a non-archimedean Banach space over \mathbb{K} . Let $A, B \in \mathcal{L}(X)$, the spectrum $\sigma(A, B)$ of a pencil of linear operator (A, B) is defined by

$$\begin{aligned}\sigma(A, B) &= \{\lambda \in \mathbb{K} : A - \lambda B \text{ is not invertible in } \mathcal{L}(X)\}, \\ &= \{\lambda \in \mathbb{K} : 0 \in \sigma(A - \lambda B)\}.\end{aligned}$$

The resolvent set $\rho(A, B)$ of a pencil of bounded linear operator (A, B) is

$$\rho(A, B) = \{\lambda \in \mathbb{K} : R(\lambda, A, B) = (A - \lambda B)^{-1} \text{ exists in } \mathcal{L}(X)\}.$$

$R(\lambda, A, B)$ is called the resolvent of pencil of bounded linear operator (A, B) .

Definition 2.2. Let X be a non-archimedean Banach space over \mathbb{K} . Let $A, B \in \mathcal{L}(X)$, the couple (A, B) is said to be regular, if $\rho(A, B) \neq \emptyset$.

For a regular couple (A, B) , we have the following definitions.

Definition 2.3. Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B \in \mathcal{L}(X)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(A, B)$ of a pencil of bounded linear operator (A, B) on X is defined by

$$\sigma_\varepsilon(A, B) = \sigma(A, B) \cup \{\lambda \in \mathbb{K} : \|(A - \lambda B)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudo-resolvent $\rho_\varepsilon(A, B)$ of a pencil of bounded linear operator (A, B) is defined by

$$\rho_\varepsilon(A, B) = \rho(A, B) \cap \{\lambda \in \mathbb{K} : \|(A - \lambda B)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(A - \lambda B)^{-1}\| = \infty$ if, and only if, $\lambda \in \sigma(A, B)$.

Definition 2.4. Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B \in \mathcal{L}(X)$ and $\varepsilon > 0$. The generalized pseudospectrum of a pencil of bounded linear operator (A, B) on X is defined by

$$\Sigma_\varepsilon(A, B) = \sigma(A, B) \cup \{\lambda \in \mathbb{K} : \|(A - \lambda B)^{-1}B\| > \varepsilon^{-1}\}.$$

By convention $\|(A - \lambda B)^{-1}B\| = \infty$ if, and only if, $\lambda \in \sigma(A, B)$.

Remark 2.5. Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B \in \mathcal{L}(X)$. Then

- (i) If $B = I$, then, $\Sigma_\varepsilon(A, I) = \sigma_\varepsilon(A)$ where $\sigma_\varepsilon(A)$ is the pseudospectrum of A .
- (ii) Definition 2.3 is a natural generalization of Definition 1.5.

In the rest of this section, we suppose that (A, B) is regular. The next proposition gives a comparison between $\sigma_\varepsilon(A, B)$ and $\Sigma_\varepsilon(A, B)$.

Proposition 2.6. Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B \in \mathcal{L}(X)$. Then for all $\varepsilon > 0$,

$$\Sigma_\varepsilon(A, B) \subset \sigma_{\varepsilon\|B\|}(A, B).$$

Proof. Let $\varepsilon > 0$ and $\lambda \in \Sigma_\varepsilon(A, B)$, then $\lambda \in \sigma(A, B)$ and

$$\frac{1}{\varepsilon} < \|(A - \lambda B)^{-1}B\| \tag{2.1}$$

$$\leq \|(A - \lambda B)^{-1}\|\|B\|. \tag{2.2}$$

Hence

$$\frac{1}{\|B\|\varepsilon} < \|(A - \lambda B)^{-1}\|.$$

Thus $\lambda \in \sigma_{\varepsilon\|B\|}(A, B)$. Consequently

$$\Sigma_\varepsilon(A, B) \subset \sigma_{\varepsilon\|B\|}(A, B).$$

□

We have the following statements.

Lemma 2.7. *Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B \in \mathcal{L}(X)$ such that $\|B\| = 1$. Then for all $\varepsilon > 0$,*

$$\Sigma_\varepsilon(A, B) \subset \sigma_\varepsilon(A, B).$$

Theorem 2.8. *Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B, C \in \mathcal{L}(X)$ such that $C^{-1} \in \mathcal{L}(X)$. Then*

(i) *For all $\varepsilon > 0$, $\Sigma_\varepsilon(A, B) = \Sigma_\varepsilon(CA, CB)$.*

(ii) *For all $\varepsilon > 0$, $\Sigma_\varepsilon(A, C) = \sigma_\varepsilon(C^{-1}A)$. In particular $C = I$, $\Sigma_\varepsilon(A, I) = \sigma_\varepsilon(A)$.*

Proof. (i) For all $\lambda \in \rho(A, B)$, we have $(A - \lambda B)^{-1}C^{-1} = (CA - \lambda CB)^{-1}$. Then, $\sigma(A, B) = \sigma(CA, CB)$. In addition, it is clear that

$$\|(CA - \lambda CB)^{-1}CB\| = \|(A - \lambda B)^{-1}B\|. \quad (2.3)$$

Hence $\lambda \in \Sigma_\varepsilon(A, B)$, if, and only if, $\lambda \in \Sigma_\varepsilon(CA, CB)$.

(ii) Assume that C is invertible, then $(A - \lambda C)^{-1}C = (C^{-1}A - \lambda I)^{-1}$. Then $\lambda \in \Sigma_\varepsilon(A, C)$, if and only if $\lambda \in \sigma_\varepsilon(C^{-1}A)$. □

Proposition 2.9. *Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B \in \mathcal{L}(X)$. For all $\varepsilon > 0$, we have*

(i) $\sigma(A, B) = \bigcap_{\varepsilon > 0} \Sigma_\varepsilon(A, B)$.

(ii) *If $0 < \varepsilon_1 \leq \varepsilon_2$, then $\sigma(A, B) \subset \Sigma_{\varepsilon_1}(A, B) \subset \Sigma_{\varepsilon_2}(A, B)$.*

Proof. (i) By Definition 2.4, we have for all $\varepsilon > 0$, $\sigma(A, B) \subset \Sigma_\varepsilon(A, B)$. Conversely, if $\lambda \in \bigcap_{\varepsilon > 0} \Sigma_\varepsilon(A, B)$, then for all $\varepsilon > 0$, $\lambda \in \Sigma_\varepsilon(A, B)$. If $\lambda \notin \sigma(A, B)$, then $\lambda \in \{\lambda \in \mathbb{K} : \|(A - \lambda B)^{-1}B\| > \varepsilon^{-1}\}$, taking limits as $\varepsilon \rightarrow 0^+$, we get $\|(A - \lambda B)^{-1}B\| = \infty$. Thus $\lambda \in \sigma(A, B)$.

(ii) For $0 < \varepsilon_1 \leq \varepsilon_2$. Let $\lambda \in \sigma_{\varepsilon_1}(A, B)$, then $\|(A - \lambda B)^{-1}B\| > \varepsilon_1^{-1} \geq \varepsilon_2^{-1}$. Hence $\lambda \in \sigma_{\varepsilon_2}(A, B)$. □

Proposition 2.10. *Let X be a non-archimedean Banach space over \mathbb{K} , let X be a free Banach space over \mathbb{K} and let $A \in \mathcal{L}(X)$ be analytic operator with compact spectrum $\sigma(A) \neq \emptyset$, then there exists $B, C \in \mathcal{L}(X)$ such that $\sigma(A) = \sigma(B, C)$. In addition we have, for all $\varepsilon > 0$, $\Sigma_\varepsilon(B, C) = \sigma_\varepsilon(A)$.*

Proof. Let $\alpha \in \rho(A)$, we set $C = (A - \alpha I)^{-1}$ and $B = A(A - \alpha I)^{-1}$. Then

$$\begin{aligned} \lambda \in \rho(A) &\iff (A - \lambda I)^{-1} \in \mathcal{L}(X) \\ &\iff (A - \lambda I)(A - \alpha I)^{-1} \in \mathcal{L}(X) \\ &\iff A(A - \alpha I)^{-1} - \lambda(A - \alpha I)^{-1} \in \mathcal{L}(X) \\ &\iff B - \lambda C \in \mathcal{L}(X) \\ &\iff (B - \lambda C)^{-1} \in \mathcal{L}(X) \\ &\iff \lambda \in \rho(B, C). \end{aligned}$$

Thus, $\sigma(A) = \sigma(B, C)$, let $\varepsilon > 0$, $z \in \sigma_\varepsilon(A)$, then $z \in \sigma(A)$ and

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(A - zI)^{-1}\|, \\ &= \|(B - zC)^{-1}C\|. \end{aligned}$$

Thus $\sigma_\varepsilon(A) = \Sigma_\varepsilon(B, C)$. □

Proposition 2.11. *Let X be a non-archimedean Banach space over \mathbb{K} and let $A \in \mathcal{C}(X)$ with $\rho(A) \neq \emptyset$, then there exists $B, C \in \mathcal{L}(X)$ such that $\sigma(A) = \sigma(B, C)$.*

Proof. Similar to the proof of Proposition 2.10. □

We have the following examples.

Example 2.12. *Let $\mathbb{K} = \mathbb{Q}_p$. Let 2×2 square matrix A and B over $\mathbb{Q}_p \times \mathbb{Q}_p$ and $a, b, c, d \in \mathbb{Q}_p^*$ such that $a \neq b$ and $c \neq d$. Then:*

(i) If

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that, for all $\lambda \in \mathbb{Q}_p$, $\det(A - \lambda B) = -\lambda a$, then $\sigma(A, B) = \{0\}$. Simple calculation, we get

$$(A - \lambda B)^{-1}B = \begin{pmatrix} \frac{-1}{\lambda} & 0 \\ 0 & 0 \end{pmatrix},$$

thus, for all $\varepsilon > 0$, $\Sigma_\varepsilon(A, B) = \{0\} \cup \{\lambda \in \mathbb{Q}_p : |\lambda|_p < \varepsilon\}$.

(ii) If

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}.$$

Note that, for all $\lambda \in \mathbb{Q}_p$, $\det(A - \lambda B) = (a - \lambda c)(b - \lambda d)$, then $\sigma(A, B) = \{\frac{a}{c}, \frac{b}{d}\}$ and

$$\|(A - \lambda B)^{-1}B\| = \max \left\{ \frac{|c|_p}{|a - \lambda c|_p}, \frac{|d|_p}{|b - \lambda d|_p} \right\}.$$

Hence, the generalized pseudospectrum of (A, B) is

$$\Sigma_\varepsilon(A, B) = \left\{ \frac{a}{c}, \frac{b}{d} \right\} \cup \left\{ \lambda \in \mathbb{Q}_p : \max \left\{ \frac{1}{|ac^{-1} - \lambda|_p}, \frac{1}{|bd^{-1} - \lambda|_p} \right\} > \frac{1}{\varepsilon} \right\}.$$

Example 2.13. *Let $A, B \in \mathcal{L}(\mathbb{E}_\omega)$ be two diagonal operators such that for all $i \in \mathbb{N}$, $Ae_i = a_i e_i$ and $Be_i = b_i e_i$, where $a_i, b_i \in \mathbb{Q}_p$ and B is invertible and $\sup_{i \in \mathbb{N}} |a_i|_p$ and $\sup_{i \in \mathbb{N}} |b_i|_p$ are finite and $0 < \inf_{i \in \mathbb{N}} |b_i|_p \leq \sup_{i \in \mathbb{N}} |b_i|_p \leq 1$. It is easy to see that*

$$\sigma(A, B) = \{ \lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i - \lambda b_i| = 0 \} = \{ \lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda| = 0 \}$$

and for all $\lambda \in \rho(A, B)$, we have

$$\begin{aligned} \|(A - \lambda B)^{-1}B\| &= \sup_{i \in \mathbb{N}} \frac{\|(A - \lambda B)^{-1}B e_i\|}{\|e_i\|} \\ &= \sup_{i \in \mathbb{N}} \left| \frac{b_i}{a_i - \lambda b_i} \right| \\ &= \frac{1}{\inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda|}. \end{aligned}$$

Hence,

$$\left\{ \lambda \in \mathbb{Q}_p : \|(A - \lambda B)^{-1}B\| > \frac{1}{\varepsilon} \right\} = \left\{ \lambda \in \mathbb{Q}_p : \frac{1}{\inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda|} > \frac{1}{\varepsilon} \right\}.$$

Consequently,

$$\Sigma_\varepsilon(A, B) = \{ \lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda| = 0 \} \cup \left\{ \lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda| < \varepsilon \right\}.$$

We have the following results.

Theorem 2.14. *Let X, Y be two non-Archimedean Banach spaces over \mathbb{K} , let $A \in \mathcal{C}(X, Y)$, $B \in \mathcal{L}(X, Y)$ and $\lambda \in \mathbb{K}$. If $A - \lambda B$ is one-to-one and onto, then $(A - \lambda B)^{-1}$ is closed linear operator.*

Proof. Let $\lambda \in \mathbb{K}$ and a sequence $(y_n)_n \subset Y$ such that y_n converges to y in Y and $(A - \lambda B)^{-1}y_n$ converges to x in X . Setting $x_n = (A - \lambda B)^{-1}y_n$, then $x_n \in D(A)$ and $y_n = (A - \lambda B)x_n \in Y$. Since $(A - \lambda B)x_n \rightarrow y$ and $x_n \rightarrow x$ and $A - \lambda B$ is a closed linear operator which implies that $x \in D(A)$ and $y = (A - \lambda B)x$, then $y \in Y$ and $x = (A - \lambda B)^{-1}y$. Thus $(A - \lambda B)^{-1}$ is closed linear operator. \square

Corollary 2.15. *Let X, Y be two non-Archimedean Banach spaces over \mathbb{K} . Let A be a linear operator from X into Y and B be a non null bounded linear operator from X into Y . If A is a non closed operator, then $\sigma(A, B) = \mathbb{K}$.*

Proof. Let A be a linear operator which is not closed. We argue by contradiction. Suppose that $\rho(A, B)$ is not empty, then there exists $\lambda \in \mathbb{K}$ such that $\lambda \in \rho(A, B)$, consequently, $(A - \lambda B)^{-1}$ is a bounded operator. Hence, $A - \lambda B$ is a closed operator. In addition, we can write $A = A - \lambda B + \lambda B$. We conclude that A is a closed operator, which is a contradiction. \square

Corollary 2.16. *Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B \in \mathcal{L}(X)$ such that $\sigma(A, B) = \mathbb{K}$, then A is not invertible.*

Proposition 2.17. *Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B \in \mathcal{L}(X)$ and $B^{-1} \in \mathcal{L}(X)$, then $\sigma_p(B^{-1}A) = \sigma_p(A, B)$.*

Proof. Let $\lambda \in \sigma_p(A, B)$, then there exists $x \in X \setminus \{0\}$ such that $Ax = \lambda Bx$. Since $B^{-1} \in \mathcal{L}(X)$, we have $B^{-1}Ax = \lambda x$, thus $\lambda \in \sigma_p(B^{-1}A)$, therefore $\sigma_p(A, B) \subseteq \sigma_p(B^{-1}A)$. Similarly, we obtain $\sigma_p(B^{-1}A) \subseteq \sigma_p(A, B)$. Thus $\sigma_p(A, B) = \sigma_p(B^{-1}A)$. \square

Theorem 2.18. *Let $A, B \in \mathcal{L}(\mathbb{K}^n)$. If A is invertible and $A^{-1}B$ or BA^{-1} is nilpotent, then $\sigma(A, B) = \emptyset$.*

Proof. Assume that A is invertible and $A^{-1}B$ or BA^{-1} is nilpotent, then for all $\lambda \in \mathbb{K}$, $I - \lambda A^{-1}B$ or $I - \lambda BA^{-1}$ is invertible, hence for all $\lambda \in \mathbb{K}$, $(A - \lambda B)^{-1}$ exists in $\mathcal{L}(\mathbb{K}^n)$. Thus, $\sigma(A, B) = \emptyset$. \square

Theorem 2.19. *Let X be a Banach space of countable type over \mathbb{Q}_p , let $A, B \in \mathcal{L}(X)$ such that B majorizes A and B is not invertible, then $\sigma(A, B) = \mathbb{Q}_p$.*

Proof. If B majorizes A . From Theorem 1.4, there exists a continuous linear operator $D : R(B) \rightarrow X$ such that $A = DB$. then, for all $\lambda \in \mathbb{Q}_p$, $A - \lambda B = (D - \lambda)B$, since B is not invertible then, for all $\lambda \in \mathbb{Q}_p$, $A - \lambda B$ is not invertible. Thus, $\sigma(A, B) = \mathbb{Q}_p$. \square

Proposition 2.20. *Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B \in \mathcal{L}(X)$ such that $AB = BA$ and $0 \in \rho(A) \cap \rho(B)$, then $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\lambda} \in \sigma(A^{-1}, B^{-1})$.*

Proof. From $A - \lambda B = -\lambda B(A^{-1} - \lambda^{-1}B^{-1})A$, we obtain that $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\lambda} \in \sigma(A^{-1}, B^{-1})$. \square

Proposition 2.21. *Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B \in \mathcal{L}(X)$ such that $AB = BA$ with $\sigma(A, B) \neq \emptyset$. If $\mu \in \rho(A, B)$, then $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\lambda - \mu} \in \sigma((A - \mu B)^{-1}B)$.*

Proof. Let $A, B \in \mathcal{L}(X)$ and $\mu \in \rho(A, B)$. For $\lambda \in \mathbb{K}$ with $\lambda \neq \mu$, we have

$$A - \lambda B = (A - \mu B)[(A - \mu B)^{-1}B - (\lambda - \mu)^{-1}](\mu - \lambda).$$

Hence $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\lambda - \mu} \in \sigma((A - \mu B)^{-1}B)$. \square

3. Non-archimedean generalized spectrum approximation

In [1], the authors extended the following definitions to non-archimedean case.

Definition 3.1. [1] Let X be a non-archimedean Banach space over \mathbb{K} and let $A \in \mathcal{L}(X)$.

- (1) A sequence (A_n) of bounded linear operators on X is said to be norm convergent to A , denoted by $A_n \rightarrow A$, if $\lim_{n \rightarrow \infty} \|A_n - A\| = 0$.
- (2) A sequence (A_n) of bounded linear operators on X is said to be pointwise convergent to A , denoted by $A_n \xrightarrow{p} A$, if for all $x \in X$, $\lim_{n \rightarrow \infty} \|A_n x - Ax\| = 0$.

Definition 3.2. [1] Let X be a non-archimedean Banach space over \mathbb{K} and let $A \in \mathcal{L}(X)$. A sequence (A_n) of bounded linear operators on X is said to be ν -convergent to A , denoted by $A_n \xrightarrow{\nu} A$, if

- (1) $(\|A_n\|)$ is bounded,
- (2) $\|(A_n - A)A\| \rightarrow 0$ as $n \rightarrow \infty$, and
- (3) $\|(A_n - A)A_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Definition 3.3. [1] Let X be a non-archimedean Banach space over a locally compact field \mathbb{K} and let $A \in \mathcal{L}(X)$. A sequence (A_n) of bounded linear operators on X is said to be convergent to A in the collectively compact convergence, denoted by $A_n \xrightarrow{c.c} A$, if $A_n \xrightarrow{p} A$, and for some positive integer N ,

$$\bigcup_{n \geq N} \{(A_n - A)x : x \in X, \|x\| \leq 1\}$$

has compact closure of X .

We have the following results.

Proposition 3.4. Let X be a non-archimedean Banach space over \mathbb{K} , let $A, A_n, B, B_n \in \mathcal{L}(X)$. If $A_n \rightarrow A$ or $B_n \rightarrow B$, then for any $C \in B(X)$, we have

$$\|(A_n - A)C(B_n - B)\| \rightarrow 0.$$

Proof. Since $A_n \rightarrow A$ or $B_n \rightarrow B$, then for any $C \in B(X)$, we have

$$\|(A_n - A)C(B_n - B)\| \leq \|(A_n - A)\| \|C\| \|B_n - B\| \rightarrow 0.$$

□

Proposition 3.5. Let X be a non-archimedean Banach space over \mathbb{K} , let $A, A_n, B, B_n \in \mathcal{L}(X)$. If $A_n \xrightarrow{\nu} A$, $B_n \xrightarrow{\nu} B$ and $0 \in \rho(A) \cap \rho(B)$, then for all $\lambda \in \mathbb{K}$, we have

$$A_n - \lambda B_n \rightarrow A - \lambda B.$$

Proof. Suppose that $A_n \xrightarrow{\nu} A$, $B_n \xrightarrow{\nu} B$, $0 \in \rho(A) \cap \rho(B)$, and for all $\lambda \in \mathbb{K}$, we have

$$\begin{aligned} \|(A_n - \lambda B_n) - (A - \lambda B)\| &\leq \max \left\{ \|(A_n - A)\|; |\lambda| \|B_n - B\| \right\} \\ &\rightarrow 0. \end{aligned}$$

Since,

$$\begin{aligned} \|(A_n - A)\| &= \|(A_n - A)AA^{-1}\| \\ &\leq \|(A_n - A)A\| \|A^{-1}\| \\ &\rightarrow 0. \end{aligned}$$

□

Similarly, we obtain $\|(B_n - B)\| \rightarrow 0$.

Proposition 3.6. *Let X be a non-archimedean Banach space over \mathbb{Q}_p such that $\|X\| \subseteq |\mathbb{Q}_p|$, let $A, A_n, B, B_n \in \mathcal{L}(X)$. If $A_n \xrightarrow{p} A$ and $B_n \xrightarrow{cc} B$, then for any $C \in \mathcal{L}(X)$, we have*

$$\|(A_n - A)C(B_n - B)\| \rightarrow 0.$$

Proof. Since $A_n \xrightarrow{p} A$ and $B_n \xrightarrow{cc} B$, and $\|X\| \subseteq |\mathbb{Q}_p|$, hence $A_n \xrightarrow{p} A$ and $B_n \xrightarrow{p} B$ and $C\left(\bigcup_{n \geq N} \{(B_n - B)x : x \in X, \|x\| \leq 1\}\right)$ has compact closure of X . Then

$$\|(A_n - A)C(B_n - B)\| \rightarrow 0.$$

□

The aim of the following results is to discuss the spectrum of a sequence of a pencil of linear operators in a non-archimedean Banach space.

Theorem 3.7. *Let X be a non-archimedean Banach space over \mathbb{K} , let (A_n) be a sequence of bounded linear operators on X and $A \in \mathcal{L}(X)$. If $A_n \rightarrow A$, then there exists $N \in \mathbb{N}$, we have*

$$\text{for all } n \geq N, \sigma(A_n) \subset \sigma(A).$$

Proof. Let $\lambda \in \rho(A)$. Then for all $n \in \mathbb{N}$, we have

$$\lambda I - A_n = (\lambda I - A)\left(I + (\lambda I - A)^{-1}(A - A_n)\right).$$

Since $A_n \rightarrow A$ Then, $\lim_{n \rightarrow \infty} \|A_n - A\| = 0$, hence for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|A_n - A\| < \varepsilon$. In particular, for $\varepsilon = \|(\lambda - A)^{-1}\|^{-1}$, we have

$$\text{for all } n \geq N, \|A_n - A\| < \|(\lambda - A)^{-1}\|^{-1}.$$

Thus, for all $n \geq N$, we have

$$\begin{aligned} \|(\lambda I - A)^{-1}(A - A_n)\| &\leq \|(\lambda I - A)^{-1}\| \|A - A_n\| \\ &< 1. \end{aligned}$$

Then for all $n \geq N$, $\left(I + (\lambda I - A)^{-1}(A - A_n)\right)^{-1} \in \mathcal{L}(X)$, hence for all $n \geq N$, $(\lambda - A_n)^{-1} \in \mathcal{L}(X)$. Thus, for all $n \geq N$, $\lambda \in \rho(A_n)$. □

We have the following proposition.

Proposition 3.8. *Let X be a non-archimedean Banach space over \mathbb{K} , let $A, B, C, D \in \mathcal{L}(X)$ such that $\rho(A, C) \neq \emptyset$. For all $z \in \rho(A, C)$ such that $\|R(z, A, C)[(A - B) - z(C - D)]\| < 1$, we have $z \in \rho(B, D)$.*

Proof. Since, for all $z \in \rho(A, C)$ such that $\|R(z, A, C)[(A - B) - z(C - D)]\| < 1$ and

$$B - zD = (A - zC)\left[I - R(z, A, C)\left((A - B) - z(C - D)\right)\right].$$

Hence $(B - zD)$ is invertible and $(B - zD)^{-1} \in \mathcal{L}(X)$. Thus, $z \in \rho(B, D)$. □

Theorem 3.9. *Let X be a non-archimedean Banach space over \mathbb{K} , let (A_n) and (B_n) be a sequences of bounded linear operators on X and $A, B \in \mathcal{L}(X)$. If $A_n \rightarrow A$ and $B_n \rightarrow B$, then there exists $N \in \mathbb{N}$, we have*

$$\text{for all } n \geq N, \sigma(A_n, B_n) \subset \sigma(A, B).$$

Proof. Let $\lambda \in \rho(A, B)$. Then for all $n \in \mathbb{N}$, we can write

$$A_n - \lambda B_n = \left(I - (E_n - \lambda F_n) \right) (A - \lambda B),$$

where $E_n = (A - A_n)R(\lambda, A, B)$ and $F_n = (B - B_n)R(\lambda, A, B)$. Since $A_n \rightarrow A$ and $B_n \rightarrow B$, then for all $\lambda \in \rho(A, B)$, $\lim_{n \rightarrow \infty} \|E_n - \lambda F_n\| = 0$. Hence there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|E_n - \lambda F_n\| < 1$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$(I - (E_n - \lambda F_n))^{-1} \in \mathcal{L}(X).$$

Thus, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $A_n - \lambda B_n$ is invertible $\mathcal{L}(X)$. Then, for all $n \geq N$, $\lambda \in \rho(A_n, B_n)$. \square

Theorem 3.10. *Let X be a non-archimedean Banach space over \mathbb{K} , let (A_n) , (B_n) be a sequences of bounded linear operators on X and $A, B \in \mathcal{L}(X)$. If $A_n \rightarrow A$ and $B_n \rightarrow B$, then for all $\lambda \in \rho(A, B)$, $(A_n - \lambda B_n)^{-1} \rightarrow (A - \lambda B)^{-1}$.*

Proof. Let $\lambda \in \rho(A, B)$. Since $A_n \rightarrow A$ and $B_n \rightarrow B$, then by using Theorem 3.9, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\lambda \in \rho(A_n, B_n)$. Then for all $n \geq N$,

$$\|(A_n - \lambda B_n)^{-1} - (A - \lambda B)^{-1}\| \tag{3.1}$$

$$\begin{aligned} (3.1) &= \|(A_n - \lambda B_n)^{-1} \left((A - \lambda B) - (A_n - \lambda B_n) \right) (A - \lambda B)^{-1}\| \\ &= \|(A_n - \lambda B_n)^{-1} \left((A - A_n) - \lambda(B - B_n) \right) (A - \lambda B)^{-1}\| \\ &\leq \max \left\{ \|A - A_n\|, |\lambda| \|B - B_n\| \right\} \|(A_n - \lambda B_n)^{-1}\| \|(A - \lambda B)^{-1}\| \end{aligned}$$

Since $A_n \rightarrow A$ and $B_n \rightarrow B$, and $\|(A_n - \lambda B_n)^{-1}\| \|(A - \lambda B)^{-1}\| < \infty$. Thus,

$$\lim_{n \rightarrow \infty} \|(A_n - \lambda B_n)^{-1} - (A - \lambda B)^{-1}\| = 0.$$

\square

Proposition 3.11. *Let X be a non-archimedean Banach space over \mathbb{K} , let (A_n) be a sequence of bounded linear operators on X and $A \in \mathcal{L}(X)$. If $A_n \rightarrow A$, then $A_n \xrightarrow{\nu} A$.*

Proof. Ovbious. \square

Theorem 3.12. *Let X be a non-archimedean Banach space over \mathbb{K} , let (A_n) , (B_n) be a sequence of bounded linear operators on X and $A, B \in \mathcal{L}(X)$. If $A_n \rightarrow A$ and $B_n \rightarrow B$, then $(A_n - \lambda B_n)^{-1} \xrightarrow{\nu} (A - \lambda B)^{-1}$ for all $\lambda \in \rho(A, B)$.*

Proof. It suffices to apply Theorem 3.10 and Proposition 3.11. \square

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