Theorems on Analogous of Ramanujan’s Remarkable Product of Theta-Function and Their Explicit Evaluations

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Abstract: In this article, we define $E_{m,n}$ for any positive real numbers $m$ and $n$ involving Ramanujan’s product of theta-functions $\psi(-q)$ and $f(q)$, which is analogous to Ramanujan’s remarkable product of theta-functions and establish its several properties by Ramanujan. We establish general theorems for the explicit evaluations of $E_{m,n}$ and its explicit values.

Key Words: Class invariant, Modular equation, Theta-function, Cubic continued fraction.

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1. Introduction

Ramanujan’s general theta-function [15] $f(a, b)$ is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab| < 1,$$

$$= (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}. \quad (1.1)$$

Three special cases of $f(a, b)$ are as follows:

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^{2}} = \frac{(-q; -q)_{\infty}}{(q; -q)_{\infty}} \quad (1.3)$$

$$\psi(q) := f(q, q^{3}) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^{2}; q^{2})_{\infty}}{(q; q^{2})_{\infty}} \quad (1.4)$$

$$f(-q) := f(-q, -q^{2}) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q; q)_{\infty}. \quad (1.5)$$

2010 Mathematics Subject Classification: 11B65, 11A55, 33D10, 11F20, 11F27 Secondary 11F27.
where

\[(a; q)_\infty := \prod_{n=0}^{\infty} (1 - a q^n), \quad |q| < 1.\]

On page 338 in his first notebook [4,15], Ramanujan defines

\[a_{m,n} = \frac{ne^{-\frac{(n-1)\pi}{4}\sqrt{mn}}}{\psi(e^{-\pi\sqrt{mn}})\varphi(e^{-2\pi\sqrt{mn}})}. \quad (1.6)\]

He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [5]. M. S. Mahadeva Naika and B. N. Dharmendra [7], also established some general theorems for explicit evaluations of the product of \(a_{m,n}\) and found some new explicit values from it. Further results on \(a_{m,n}\) are found by Mahadeva Naika, Dharmendra and K. Shivashankara [9], and Mahadeva Naika and M. C. Mahesh Kumar [10]. Recently Nipen Saikia [13] established new properties of \(a_{m,n}\).

In [12], Mahadeva Naika et al. defined the product

\[b_{m,n} = \frac{ne^{-\frac{(n-1)\pi}{4}\sqrt{mn}}}{\psi(e^{-\pi\sqrt{mn}})\varphi(e^{-2\pi\sqrt{mn}})}. \quad (1.7)\]

They established general theorems for explicit evaluation of \(b_{m,n}\) and obtained some particular values. Mahadeva Naika et al. [11] established general formulas for explicit values of Ramanujan’s cubic continued fraction \(V(q)\) in terms of the products of \(a_{m,n}\) and \(b_{m,n}\) defined above, where

\[V(q) := \frac{q^{1/3}}{1} + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \frac{q^3 + q^6}{1} + \cdots, \quad |q| < 1, \quad (1.8)\]

and found some particular values of \(V(q)\).

In this paper, we define

\[E_{m,n} = \frac{f(e^{-\pi\sqrt{mn}})}{\psi(e^{-\pi\sqrt{mn}})\varphi(e^{-2\pi\sqrt{mn}})}. \quad (1.9)\]

where \(m\) and \(n\) are positive real numbers.

Let \(K, K', L\) and \(L'\) denote the complete elliptic integrals of the first kind associated with the moduli \(k, k' := \sqrt{1-k^2}, l\) and \(l' := \sqrt{1-l^2}\) respectively, where 0 < \(k, l < 1\). For a fixed positive integer \(n\), suppose that

\[\frac{K'}{K} = \frac{L'}{L}. \quad (1.10)\]

Then a modular equation of degree \(n\) is a relation between \(k\) and \(l\) induced by (1.5). Following Ramanujan, set \(\alpha = k^2\) and \(\beta = l^2\). Then we say \(\beta\) is of degree \(n\) over \(\alpha\).

Define

\[\chi(q) := (-q; q^2)_\infty.\]
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and

\[ G_\alpha := 2^{\alpha} q \cdot \chi(q), \]

where

\[ q = e^{-\pi \sqrt{T}}. \]

Moreover, if \( q = e^{-\pi \sqrt{\frac{m}{n}}} \) and \( \beta \) has degree \( n \) over \( \alpha \), then

\[ G_{mn} = (4\alpha(1-\alpha))^{\frac{1}{24}} \]

and

\[ G_{nm} = (4\beta(1-\beta))^{\frac{1}{24}}. \]

The main purpose of this paper is to obtain several general theorems for the explicit evaluations of analogous of Ramanujan’s product of theta-function \( E_{m,n} \) and also some new explicit evaluations from it.

2. Preliminary Results

In this section, we collect several identities which are useful in proving our main results.

Lemma 2.1. \([2, \text{Ch. 17, Entry 11(ii) and Entry 12(i), pp. 123–124}]\) We have,

\[ 2^{1/2} e^{-y/8} \psi(-e^{-y}) = \sqrt{\alpha} \{\alpha(1-\alpha)\}^{1/8}, \]

\[ 2^{1/2} e^{-my/8} \psi(-e^{-my}) = \sqrt{\beta} \{\beta(1-\beta)\}^{1/8}, \]

\[ 2^{1/6} e^{-y/24} f(e^{-y}) = \sqrt{\alpha} \{\alpha(1-\alpha)\}^{1/24}, \]

\[ 2^{1/6} e^{-my/24} f(e^{-my}) = \sqrt{\beta} \{\beta(1-\beta)\}^{1/24}. \]

Lemma 2.2. \([2, \text{Ch. 16, Entry 27(iii) and (iv), pp. 43}]\) We have,

\[ e^{-\alpha/24} \sqrt{\alpha} f(-e^{-\alpha}) = e^{-\beta/24} \sqrt{\beta} f(-e^{-\beta}), \text{ if } \alpha \beta = \pi^2 \]

\[ e^{-\alpha/12} \sqrt{\alpha} f(-e^{-2\alpha}) = e^{-\beta/12} \sqrt{\beta} f(-e^{-2\beta}), \text{ if } \alpha \beta = \pi^2. \]

Lemma 2.3. \([6, \text{Theorem 2.1}]\) We have,

\[ \frac{f^6(q)}{f^6(q^3)} = \frac{\psi^2(-q)}{\psi^2(-q^3)} \left\{ \frac{\psi^4(-q) + 9\psi^4(-q^3)}{\psi^4(-q) + q\psi^4(-q^3)} \right\}. \]

Lemma 2.4. \([16, \text{p. 56}]\) \([14]\) We have,

\[ \frac{f^3(q)}{f^3(q^5)} = \frac{\psi(-q)}{\psi(-q^5)} \left\{ \frac{\psi^2(-q) + 5q\psi^2(-q^5)}{\psi^2(-q) + q\psi^2(-q^5)} \right\}. \]

Lemma 2.5. \([6, \text{Theorem 2.2}]\) We have,

\[ \frac{f^3(q)}{f^3(q^5)} = \frac{\psi(-q)}{\psi(-q^5)} \left\{ \frac{\psi(-q) + 3q\psi(-q^5)}{\psi(-q) + q\psi(-q^5)} \right\}^2. \]
Lemma 2.6. [2, Chapter 19, entry 5(xii), page 231] We have, If \( P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \) and \( Q := \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/4} \), then
\[
Q + \frac{1}{Q} = 2\sqrt{2} \left( \frac{1}{P} - P \right).
\] (2.10)

Lemma 2.7. [2, Chapter 19, entry 13(xiv), page 282] We have, If \( P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \) and \( Q := \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/8} \), then
\[
Q + \frac{1}{Q} = 2 \left( \frac{1}{P} - P \right).
\] (2.11)

Lemma 2.8. [2, Chapter 19, entry 19(ix), page 315] We have, If \( P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \) and \( Q := \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/6} \), then
\[
Q + \frac{1}{Q} + 7 = 2\sqrt{2} \left( P + \frac{1}{P} \right).
\] (2.12)

Lemma 2.9. [1, Theorem 5.1] We have, If \( P = \frac{\psi(-q)}{q^{1/4}\psi(-q^{y})} \) and \( Q = \frac{\varphi(q)}{\varphi(q^{y})} \), then
\[
Q^4 + P^4Q^4 = 9 + P^4.
\] (2.13)

Lemma 2.10. [1, Theorem 5.1] We have, If \( P = \frac{\psi(-q)}{q^{1/2}\psi(-q^{y})} \) and \( Q = \frac{\varphi(q)}{\varphi(q^{y})} \), then
\[
Q^2 + P^2Q^2 = 5 + P^2.
\] (2.14)

Lemma 2.11. [8, Theorem 3.2] We have, If \( P = \frac{\psi(-q)}{q\psi(-q^{y})} \) and \( Q = \frac{\varphi(q)}{\varphi(q^{y})} \), then
\[
Q + PQ = 3 + P.
\] (2.15)

3. Some Properties of \( E_{m,n} \)

In this section, we have establish some properties of \( E_{m,n} \).

Theorem 3.1.
\[
E_{m,n} = E_{n,m}.
\] (3.1)

Proof. Employing the equation (2.5) and (2.6), we deduce that
\[
e^{-\alpha/8} \sqrt{\alpha} \psi(-e^{-\alpha}) = e^{-\beta/8} \sqrt{\beta} \psi(-e^{-\beta}), \text{ if } \alpha \beta = \pi^2.
\] (3.2)

Using the equations (2.5) and (3.2) in (1.9), we obtain (3.1). \( \square \)
Theorem 3.2. \[ E_{m,n}E_{m,1/n} = 1. \] \[ (3.3) \]

Proof. Using the equations (2.5) and (3.2) in (1.9), we obtain (3.3). \[ \Box \]

Corollary 3.3. \[ E_{m,1} = 1. \] \[ (3.4) \]

Proof. Putting \( n = 1 \) in the equation (3.3), we get (3.4). \[ \Box \]

Remark 3.4. By using the definition of \( \psi(q) \), \( f(q) \) and \( E_{m,n} \), it can be seen that \( E_{m,n} \) has positive real value and that the values of \( E_{m,n} \) increases as \( n \) increase when \( m > 1 \). Thus by the above corollary, \( E_{m,n} > 1 \) for all \( n > 1 \) if \( m > 1 \).

Theorem 3.5. \[ \frac{E_{km,n}}{E_{nm,k}} = E_{m,n}^{k/m}. \] \[ (3.5) \]

Proof. Employing the definition of \( E_{m,n} \), we obtain
\[ \frac{E_{km,n}}{E_{nm,k}} = e^{\frac{\pi}{12}(\sqrt{mn} - \sqrt{nm})} \frac{f(e^{-\pi \sqrt{mn}}) \psi(-e^{\pi \sqrt{mn}})}{f(e^{-\pi \sqrt{nm}}) \psi(-e^{\pi \sqrt{nm}})}. \] \[ (3.6) \]
Using the Lemma 2.2 in the above equation (3.6) and simplifying using the Theorems 3.1 and 3.2, we obtain (3.5). \[ \Box \]

Corollary 3.6. \[ E_{m^2,n} = E_{nm,n}E_{m,\frac{n}{m}}. \] \[ (3.7) \]

Proof. Putting \( m = n \) in the above Theorem 3.5 and simplifying using the equation (3.3), we get
\[ E_{m^2,k} = E_{mk,n}E_{m,\frac{k}{m}}. \] \[ (3.8) \]
Replace \( k \) by \( n \), we obtain (3.7). \[ \Box \]

Theorem 3.7. If \( mn = rs \)
\[ \frac{E_{m,n}}{E_{kr,ks}} = \frac{E_{r,s}}{E_{km,km}}. \] \[ (3.9) \]

Proof. Using the definition of \( E_{m,n} \) and letting \( mn = rs \) for positive real numbers \( m, n, r, s \) and \( k \), we find that
\[ \frac{E_{km,km}}{E_{m,n}} = \frac{E_{kr,ks}}{E_{r,s}}. \] \[ (3.10) \]
On rearranging the above equation (3.10) we obtain the required result. \[ \Box \]
Corollary 3.8. If \( mn = rs \)

\[
E_{np,np} = E_{np^2,n}E_{p,p}. \tag{3.11}
\]

Proof. Letting \( m = p^2, \ n = 1, \ r = s = p \) and \( k = n \) in above Theorem 3.7, we deduced the equation (3.11). \( \square \)

Theorem 3.9. For all positive real numbers \( m, n, r \) and \( s \), then

\[
E_{m/n,r/s} = E_{ms/np,ns/pr}. \tag{3.12}
\]

Proof. Employing the equation (3.3) in equation (3.5), we find that, for all positive real numbers \( m, n \) and \( k \)

\[
E_{m/n,k} = E_{m,nk}E_{n/mk}. \tag{3.13}
\]

Letting \( k = r/s \) and again using the equation (3.5) and (3.1) in (3.13), we get (3.12). \( \square \)

Theorem 3.10.

\[
E_{m/n,m/n} = E_{n,n}E_{m,m/n^2}. \tag{3.14}
\]

Proof. Using the Theorems 3.2 and 3.9, we get (3.14). \( \square \)

Theorem 3.11.

\[
E_{m,m}E_{m,m^2/n} = E_{n,n}E_{n,m^2/n}. \tag{3.15}
\]

Proof. Putting \( k = m/n \) in the equation (3.13) and Employing Theorems 3.2 and 3.10, we obtain (3.15). \( \square \)

Theorem 3.12.

\[
E_{m,m}E_{n,m^2} = E_{n,n}E_{m,mn^2}. \tag{3.16}
\]

Proof. Employing the Theorems 3.1, 3.2, 3.10 and 3.11, we obtain (3.16). \( \square \)

4. Some General Theorems on \( E_{m,n} \) and their explicit evaluations

In this section we establish some general theorems and their explicit evaluations of Ramanujan’s remarkable product of theta functions involving \( E_{m,n} \).

Theorem 4.1. If \( n \) is any positive real \( P := \{G_n/3G_{3n}\}^3 \) and \( Q := E_{3,n}^3 \), then

\[
Q + \frac{1}{Q} = 2\sqrt{2}\left\{ P - \frac{1}{P} \right\}. \tag{4.1}
\]
Proof. Using the Lemma 2.1 with the definition of $E_{m,n}$, we obtain

$$E_{m,n} = \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/12}. \quad (4.2)$$

Employing the above equation (4.2) and definition of class invariant (1.11), (1.12) in the Lemma 2.6 with $m = 3$, we obtain (4.1) \(\square\)

Corollary 4.2.

$$E_{3,9} = \left\{ 1 + 2^{2/3} - 2^{4/3} \right\}^{1/3}. \quad (4.3)$$

Proof. Putting $n = 9$ in the above Theorem 4.1, we obtain

$$E_{3,3} + E_{3,9} + 7 = 2 \sqrt{2} \left\{ P + 1 \right\}. \quad (4.6)$$

Theorem 4.4. If $n$ is any positive real $P := \{G_{n/5}G_{5n}\}^3$ and $Q := E_{1,n}^{3/2}$, then

$$Q + 1 = 2 \left\{ P - \frac{1}{P} \right\}. \quad (4.5)$$

Proof. Using the equation (4.2) and definition of class invariant (1.11), (1.12) in the Lemma 2.7 with $m = 5$, we obtain (4.5). \(\square\)

Theorem 4.5.

$$E_{3,n} = \frac{f(q)\psi(-q^3)}{q^{-1/4}f(q^3)\psi(-q)}; \quad q := e^{-\pi \sqrt{n}} \quad (4.7)$$

$$E_{3,n} = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)}\right\} \quad \text{if} \quad P^4 \neq -1. \quad (4.9)$$

Theorem 4.3. If $n$ is any positive real $P := \{G_{n/7}G_{7n}\}^3$ and $Q := E_{2,n}^3$, then

$$Q + 1 = 2 \sqrt{2} \left\{ P + 1 \right\}. \quad (4.6)$$

Proof. Using the equation (4.2) and definition of class invariant (1.11), (1.12) in the Lemma 2.8 with $m = 7$, we obtain (4.6). \(\square\)
Proof. Employing the definition of $E_{m,n}$ with $m = 3$, we get
\[ E_{3,n} = \frac{f(q)\psi(-q^3)}{q^{-1/6}f(q^3)\psi(-q)}. \tag{4.10} \]
Raising the power by 6 in the above equation (4.10) with the Lemma 2.3, we deduce that
\[ E_{6,3}^6 = \frac{f^6(q)\psi^6(-q^3)}{q^{-1}f^6(q^3)\psi^6(-q)}. \tag{4.11} \]
\[ E_{6,3}^6 = \frac{P^2\left(\frac{P^4 + 9}{1 + P^2}\right)}{P^6}. \tag{4.12} \]
On simplifying the above equation (4.12), we obtain (4.9). $\square$

Corollary 4.6.
\[ E_{3,3} = \left\{2 - \sqrt{3}\right\}^{1/3}. \tag{4.13} \]
Proof. Putting $n = 3$ in the equation (4.8) and from Ramanujan’s Notebooks [4, p. 327], we have
\[ \frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = \frac{4}{\sqrt{6\sqrt{3}} - 9}. \tag{4.14} \]
Employing the equation (2.13) and (4.14), we obtain
\[ P := \frac{\psi(-e^{-\pi})}{\psi(-e^{-3\pi})} = \sqrt{9 + 6\sqrt{3}}. \tag{4.15} \]
Substituting (4.15) in (4.9), we obtain the required result. $\square$

Theorem 4.7.
\[ E_{5,n} = \frac{f(q)\psi(-q^5)}{q^{-1/3}f(q^5)\psi(-q)}, \quad q := e^{-\pi\sqrt{5}}. \tag{4.16} \]
If
\[ P := \frac{\psi(-q)}{q^{1/2}\psi(-q^3)} \quad \text{and} \quad Q := \frac{f(q)}{q^{1/6}f(q^3)}, \quad \text{then} \tag{4.17} \]
\[ E_{5,n}^3 = \frac{P^2 + 5}{P^2(P^2 + 1)}, \quad \text{if} \quad P^2 \neq -1. \tag{4.18} \]
Proof. Employing the definition of $E_{m,n}$ with $m = 5$, we get
\[ E_{5,n} = \frac{f(q)\psi(-q^5)}{q^{-1/3}f(q^5)\psi(-q)}. \tag{4.19} \]
Raising the power by 3 in the above equation (4.19) with the Lemma 2.4, we deduce that

$$E_{5,n}^3 = \frac{f^3(q)\psi^3(-q^2)}{q^{-1}f^3(q^5)\psi^3(-q)}.$$  \hspace{1cm} (4.20)

$$E_{5,n}^3 = \frac{P\left\{\frac{5 + P^2}{P^2 + 1}\right\}}{P^3}.$$  \hspace{1cm} (4.21)

On simplifying the above equation (4.21), we obtain (4.18). \hfill \Box

**Corollary 4.8.**

$$E_{5,5} = \left\{9 - 4\sqrt{5}\right\}^{2/3}.$$  \hspace{1cm} (4.22)

**Proof.** Putting $n = 5$ in the equation (4.17) and from Ramanujan’s Notebooks [4, p. 327], we have

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-5\pi})} = \sqrt{5}\sqrt{5} - 10.$$  \hspace{1cm} (4.23)

Employing the equation (2.14) and (4.23), we obtain

$$P := \frac{\psi(-e^{-\pi})}{\psi(-e^{-5\pi})} = \sqrt{5}\sqrt{5} + 10.$$  \hspace{1cm} (4.24)

Substituting (4.24) in (4.18), we obtain the required result. \hfill \Box

**Theorem 4.9.**

$$E_{9,n} = \frac{f(q)\psi(-q^3)}{q^{-2/3}f(q^3)\psi(-q)}; \hspace{0.5cm} q := e^{-\pi\sqrt{3}}.$$  \hspace{1cm} (4.25)

If

$$P := \frac{\psi(-q)}{q\psi(-q^3)} \hspace{0.5cm} \text{and} \hspace{0.5cm} Q := \frac{f(q)}{q^{1/3}f(q^3)} \hspace{1cm} \text{then}$$

$$E_{9,n}^3 = \left\{\frac{P + 3}{P(P + 1)}\right\}^2, \hspace{0.5cm} \text{if} \hspace{0.5cm} P \neq -1.$$  \hspace{1cm} (4.27)

**Proof.** Employing the definition of $E_{m,n}$ with $m = 9$, we get

$$E_{9,n} = \frac{f(q)\psi(-q^3)}{q^{-2/3}f(q^3)\psi(-q)}.$$  \hspace{1cm} (4.28)

Raising the power by 3 in the above equation (4.28) with the Lemma 2.5, we deduce that

$$E_{9,n}^3 = \frac{f^3(q)\psi^3(-q^2)}{q^{-2}f^3(q^9)\psi^3(-q)}.$$  \hspace{1cm} (4.29)

$$E_{9,n}^3 = \frac{P\left\{\frac{P + 3}{P^2 + 1}\right\}^2}{P^3}.$$  \hspace{1cm} (4.30)
On simplifying the above equation (4.30), we obtain (4.27).

Corollary 4.10.

\[ E_{9,9} = \left\{ \frac{33s^2 - (39 + \sqrt{3})s - 21\sqrt{3} + 6}{33} \left[ 54 - 31\sqrt{3} \right] \right\}^{1/3}. \]  

(4.31)

where \( s = (2\sqrt{3} + 2)^{1/3} \)

Proof. Putting \( n = 9 \) in the equation (4.26) and from Ramanujan's Notebooks [4, p. 327] we have,

\[ P := \frac{\varphi(e^{-\pi})}{\varphi(e^{-9\pi})} = \frac{3}{1 + \sqrt{2}(\sqrt{3} + 1)}. \]  

(4.32)

Employing the equation (2.15) and (4.32), we obtain

\[ P := \frac{\psi(-e^{-\pi})}{\psi(-e^{-9\pi})} = \frac{(s^2 + 2s + \sqrt{3} + 1)(3 + \sqrt{3})}{2}. \]  

(4.33)

Substituting (4.33) in (4.27), we obtain the required result.

Theorem 4.11.

\[ E_{m,n} = \left\{ \frac{G_{n/m}}{G_{mn}} \right\}^2. \]  

(4.34)

Proof. Employing the Lemma 2.1 in the definition of \( E_{m,n} \), we obtain

\[ E_{m,n} = \left\{ \frac{\beta(1 - \beta)}{\alpha(1 - \alpha)} \right\}^{1/12}. \]  

(4.35)

Using the equation (1.11) and (1.12), we get

\[ \frac{G_{nm}}{G_{n/m}} = \left\{ \frac{\alpha(1 - \alpha)}{\beta(1 - \beta)} \right\}^{1/24}. \]  

(4.36)

By observing the equations (4.35) and (4.36), we obtain (4.34).

Corollary 4.12.

\[ E_{n,n} = G_{n^2}^{-2}. \]  

(4.37)

Proof. Setting \( m = n \) in the above Theorem 4.7 with the value \( G_1 = 1 \), we obtain the required result.
Corollary 4.13.

(i) \( E_{2,2} = 2^{3/8}(1 + \sqrt{2})^{-1/2} \), \hspace{1cm} (4.38)

(ii) \( E_{3,3} = \left\{ 2 - \sqrt{3} \right\}^{1/3} \), \hspace{1cm} (4.39)

(iii) \( E_{5,5} = \frac{3 - \sqrt{5}}{2} \), \hspace{1cm} (4.40)

(iv) \( E_{9,9} = \left\{ \frac{2(\sqrt{3} + 1)}{2(\sqrt{3} - 1)} \right\}^{1/3} \). \hspace{1cm} (4.41)

Proof. For (i), we use the values of \( G_4 \) from [3, p.114, Theorem 6.2.2(ii)]. For (ii) – (iv), we use corresponding values of \( G_n \) from [2, p.189-193]. \hspace{1cm} \( \Box \)

Acknowledgments

The authors are grateful to the referee for his useful comments which considerably improves the quality of the paper.

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