Theorems on Analogous of Ramanujan’s Remarkable Product of Theta-Function and Their Explicit Evaluations

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ABSTRACT: In this article, we define $E_{m,n}$ for any positive real numbers $m$ and $n$ involving Ramanujan’s product of theta-functions $\psi(-q)$ and $f(q)$, which is analogous to Ramanujan’s remarkable product of theta-functions and establish its several properties by Ramanujan. We establish general theorems for the explicit evaluations of $E_{m,n}$ and its explicit values.

Key Words: Class invariant, Modular equation, Theta-function, Cubic continued fraction.

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1. Introduction

Ramanujan’s general theta-function [15] $f(a, b)$ is defined by

\[ f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \]
\[ = (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}. \] (1.1)

Three special cases of $f(a, b)$ are as follows:

\[ \varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}}, \] (1.3)

\[ \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, \] (1.4)

\[ f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q; q)_{\infty}, \] (1.5)

where

\[ (a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1. \]

On page 338 in his first notebook [4,15], Ramanujan defines

\[ a_{m,n} = \frac{ne^{-\frac{(n-1)\pi}{4}} \sqrt{n \pi} \psi^2(e^{-\pi \sqrt{mn}}) \varphi^2(e^{-2\pi \sqrt{mn}})}{\psi^2(e^{-\pi \sqrt{mn}}) \varphi^2(e^{-2\pi \sqrt{mn}})}. \] (1.6)

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He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [5]. M. S. Mahadeva Naika and B. N. Dharmendra [7], also established some general theorems for explicit evaluations of the product of $a_{m,n}$ and found some new explicit values from it. Further results on $a_{m,n}$ are found by Mahadeva Naika, Dharmendra and K. Shivashankara [9], and Mahadeva Naika and M. C. Mahesh Kumar [10]. Recently Nipen Saikia [13] established new properties of $a_{m,n}$.

In [12], Mahadeva Naika et al. defined the product

$$b_{m,n} = \frac{ne^{-(n-1)\pi/4} \sqrt{mn} \psi^2(-e^{-\pi \sqrt{mn}}) \varphi^2(-e^{-2\pi \sqrt{mn}})}{\psi^2(-e^{-\pi \sqrt{mn}}) \varphi^2(-e^{-2\pi \sqrt{mn}})}. \quad (1.7)$$

They established general theorems for explicit evaluation of $b_{m,n}$ and obtained some particular values. Mahadeva Naika et al. [11] established general formulas for explicit values of Ramanujan’s cubic continued fraction $V(q)$ in terms of the products of $a_{m,n}$ and $b_{m,n}$ defined above, where

$$V(q) := \frac{q^{1/3}}{1} + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \frac{q^3 + q^6}{1} + \cdots, \quad |q| < 1, \quad (1.8)$$

and found some particular values of $V(q)$.

In this paper, we define

$$E_{m,n} = \frac{f(e^{-\pi \sqrt{mn}})\psi(-e^{-\pi \sqrt{mn}})}{e^{-\pi(1+1/m)\sqrt{mn}} f(e^{-\pi \sqrt{mn}})\psi(-e^{-\pi \sqrt{mn}})}, \quad (1.9)$$

where $m$ and $n$ are positive real numbers.

Let $K, K', L$ and $L'$ denote the complete elliptic integrals of the first kind associated with the moduli $k, k':=\sqrt{1-k^2}$, $l$ and $l':=\sqrt{1-l^2}$ respectively, where $0 < k, l < 1$. For a fixed positive integer $n$, suppose that

$$n\frac{K'}{K} = \frac{L'}{L}. \quad (1.10)$$

Then a modular equation of degree $n$ is a relation between $k$ and $l$ induced by (1.5). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say $\beta$ is of degree $n$ over $\alpha$.

Define

$$\chi(q) := (-q, q^2) \infty,$$

and

$$G_n := 2^{-\frac{1}{4}} q^{-\frac{1}{4\pi}} \chi(q),$$

where

$$q = e^{-\pi \sqrt{r}}.$$ 

Moreover, if $q = e^{-\pi \sqrt{m}}$ and $\beta$ has degree $n$ over $\alpha$, then

$$G_{m} = (4\alpha(1-\alpha))^{\frac{1}{4\pi}} \quad (1.11)$$

and

$$G_{nm} = (4\beta(1-\beta))^{\frac{1}{4\pi}}. \quad (1.12)$$

The main purpose of this paper is to obtain several general theorems for the explicit evaluations of analogous of Ramanujan’s product of theta-function $E_{m,n}$ and also some new explicit evaluations from it.


\section{Preliminary Results}

In this section, we collect several identities which are useful in proving our main results.

\textbf{Lemma 2.1.} \textit{[2, Ch. 17, Entry 11(ii) and Entry 12(i), pp. 123–124]} We have,
\begin{align}
2^{1/2} e^{-y/8} \psi(-e^{-y}) &= \sqrt{z} \{ \alpha(1-\alpha) \}^{1/8}, \quad (2.1) \\
2^{1/2} e^{-my/8} \psi(-e^{-my}) &= \sqrt{zm} \{ \beta(1-\beta) \}^{1/8}, \quad (2.2) \\
2^{1/6} e^{-y/24} f(e^{-y}) &= \sqrt{z_1} \{ \alpha(1-\alpha) \}^{1/24}, \quad (2.3) \\
2^{1/6} e^{-my/24} f(e^{-my}) &= \sqrt{zm} \{ \beta(1-\beta) \}^{1/24}. \quad (2.4)
\end{align}

\textbf{Lemma 2.2.} \textit{[2, Ch. 16, Entry 27(iii) and (iv), pp. 43]} We have,
\begin{align}
e^{-\alpha/24} \sqrt{\alpha} f(e^{-\alpha}) &= e^{-\beta/24} \sqrt{\beta} f(e^{-\beta}), \quad \text{if } \alpha\beta = \pi^2 \\
e^{-\alpha/12} \sqrt{\alpha} f(-e^{-2\alpha}) &= e^{-\beta/12} \sqrt{\beta} f(-e^{-2\beta}), \quad \text{if } \alpha\beta = \pi^2. \quad (2.5)
\end{align}

\textbf{Lemma 2.3.} \textit{[6, Theorem 2.1]} We have,
\begin{equation}
\frac{f^6(q)}{f^6(q^3)} = \frac{\psi^2(q)}{\psi^2(q^3)} \left\{ \frac{\psi^4(q) + 9q\psi^4(q^3)}{\psi^4(q) + q\psi^4(q^3)} \right\}. \quad (2.7)
\end{equation}

\textbf{Lemma 2.4.} \textit{[16, p. 56]} \textit{[14]} We have,
\begin{equation}
\frac{f^3(q)}{f^3(q^3)} = \frac{\psi(q)}{\psi(q^3)} \left\{ \frac{\psi(q) + 5q\psi(q^3)}{\psi(q) + \psi(q^3)} \right\}. \quad (2.8)
\end{equation}

\textbf{Lemma 2.5.} \textit{[6, Theorem 2.2]} We have,
\begin{equation}
\frac{f^3(q)}{f^3(q^3)} = \frac{\psi(q)}{\psi(q^3)} \left\{ \frac{\psi(q) + 3q\psi(q^3)}{\psi(q) + \psi(q^3)} \right\}^2. \quad (2.9)
\end{equation}

\textbf{Lemma 2.6.} \textit{[2, Chapter 19, entry 5(xii), page 23]} We have,
If \( P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \) and \( Q := \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/4} \), then
\begin{equation}
Q + \frac{1}{Q} = 2\sqrt{2} \left( \frac{1}{P} - P \right). \quad (2.10)
\end{equation}

\textbf{Lemma 2.7.} \textit{[2, Chapter 19, entry 13(xiv), page 282]} We have,
If \( P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \) and \( Q := \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/8} \), then
\begin{equation}
Q + \frac{1}{Q} = 2 \left( \frac{1}{P} - P \right). \quad (2.11)
\end{equation}

\textbf{Lemma 2.8.} \textit{[2, Chapter 19, entry 19(ix), page 315]} We have,
If \( P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \) and \( Q := \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/6} \), then
\begin{equation}
Q + \frac{1}{Q} + 7 = 2\sqrt{2} \left( P + \frac{1}{P} \right). \quad (2.12)
\end{equation}

\textbf{Lemma 2.9.} \textit{[1, Theorem 5.1]} We have,
If \( P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \) and \( Q = \frac{\varphi(q)}{\varphi(q^3)} \), then
\begin{equation}
Q^4 + P^4Q^4 = 9 + P^4. \quad (2.13)
\end{equation}
Lemma 2.10. [1, Theorem 5.1] We have,
If \( P = \frac{\psi(-q)}{q^{1/2} \psi(-q^5)} \) and \( Q = \frac{\varphi(q)}{\varphi(q^5)} \), then
\[
Q^2 + P^2 Q^2 = 5 + P^2.
\] (2.14)

Lemma 2.11. [8, Theorem 3.2] We have,
If \( P = \frac{\psi(-q)}{q \psi(-q^9)} \) and \( Q = \frac{\varphi(q)}{\varphi(q^9)} \), then
\[
Q + PQ = 3 + P.
\] (2.15)

3. Some Properties of \( E_{m,n} \)

In this section, we have established some properties of \( E_{m,n} \).

Theorem 3.1.
\[
E_{m,n} = E_{n,m}.
\] (3.1)

Proof. Employing the equation (2.5) and (2.6), we deduce that
\[
e^{-\alpha/8} \sqrt{\alpha} \psi(-e^{-\alpha}) = e^{-\beta/8} \sqrt{\beta} \psi(-e^{-\beta}), \quad \text{if } \alpha \beta = \pi^2.
\] (3.2)

Using the equations (2.5) and (3.2) in (1.9), we obtain (3.1). \( \square \)

Theorem 3.2.
\[
E_{m,n} E_{m,1/n} = 1.
\] (3.3)

Proof. Using the equations (2.5) and (3.2) in (1.9), we obtain (3.3). \( \square \)

Corollary 3.3.
\[
E_{m,1} = 1.
\] (3.4)

Proof. Putting \( n = 1 \) in the equation (3.3), we get (3.4). \( \square \)

Remark 3.4. By using the definition of \( \psi(q) \), \( f(q) \) and \( E_{m,n} \), it can be seen that \( E_{m,n} \) has positive real value and that the values of \( E_{m,n} \) increases as \( n \) increase when \( m > 1 \). Thus by the above corollary, \( E_{m,n} > 1 \) for all \( n > 1 \) if \( m > 1 \).

Theorem 3.5.
\[
\frac{E_{km,n}}{E_{nm,k}} = E_{m, \frac{n}{k}}.
\] (3.5)

Proof. Employing the definition of \( E_{m,n} \), we obtain
\[
\frac{E_{km,n}}{E_{nm,k}} = e^{\sqrt{\frac{\pi}{mn}} e^{-\pi \sqrt{\frac{1}{nm}}}} \frac{f(e^{-\pi \sqrt{\frac{1}{mn}}}) \psi(-e^{-\pi \sqrt{\frac{1}{mn}}})}{f(e^{-\pi \sqrt{\frac{1}{mn}}}) \psi(-e^{-\pi \sqrt{\frac{1}{mn}}})}.
\] (3.6)

Using the Lemma 2.2 in the above equation (3.6) and simplifying using the Theorems 3.1 and 3.2, we obtain (3.5). \( \square \)

Corollary 3.6.
\[
E_{m^2,n} = E_{nm,n} E_{m, \frac{n}{m}}.
\] (3.7)

Proof. Putting \( m = n \) in the above Theorem 3.5 and simplifying using the equation (3.3), we get
\[
E_{m^2,k} = E_{mk,n} E_{m, \frac{k}{m}}.
\] (3.8)

Replace \( k \) by \( n \), we obtain (3.7). \( \square \)
Theorem 3.7. If $mn = rs$

\[ \frac{E_{m,n}}{E_{kr,ks}} = \frac{E_{r,s}}{E_{km,kn}}. \quad (3.9) \]

Proof. Using the definition of $E_{m,n}$ and letting $mn = rs$ for positive real numbers $m, n, r, s$ and $k$, we find that

\[ \frac{E_{km,kn}}{E_{m,n}} = \frac{E_{kr,ks}}{E_{r,s}}. \quad (3.10) \]

On rearranging the above equation (3.10) we obtain the required result.

\[ \square \]

Corollary 3.8. If $mn = rs$

\[ E_{np,np} = E_{np^2,n}E_{p,p}. \quad (3.11) \]

Proof. Letting $m = p^2$, $n = 1$, $r = s = p$ and $k = n$ in above Theorem 3.7, we deduced the equation (3.11).

\[ \square \]

Theorem 3.9. For all positive real numbers $m, n, r$ and $s$, then

\[ E_{m/n,r/s} = E_{ms,nr}E_{mr,ns}. \quad (3.12) \]

Proof. Employing the equation (3.3) in equation (3.5), we find that, for all positive real numbers $m, n$ and $k$

\[ E_{m/n,k} = E_{m,nk}E_{n,mk}^{-1}. \quad (3.13) \]

Letting $k = r/s$ and again using the equation (3.5) and (3.1) in (3.13), we get (3.12).

\[ \square \]

Theorem 3.10.

\[ E_{m/m,n/n} = E_{n,n}E_{m,m/n^2}. \quad (3.14) \]

Proof. Using the Theorems 3.2 and 3.9, we get (3.14).

\[ \square \]

Theorem 3.11.

\[ E_{m,n}E_{m^2/m} = E_{n,n}E_{n,m^2/n}. \quad (3.15) \]

Proof. Putting $k = m/n$ in the equation (3.13) and Employing Theorems 3.2 and 3.10, we obtain (3.15).

\[ \square \]

Theorem 3.12.

\[ E_{m,m}E_{n,m^2} = E_{n,n}E_{m,mn^2}. \quad (3.16) \]

Proof. Employing the Theorems 3.1, 3.2, 3.10 and 3.11, we obtain (3.16).

\[ \square \]

4. Some General Theorems on $E_{m,n}$ and their explicit evaluations

In this section we establish some general theorems and their explicit evaluations of Ramanujan’s remarkable product of theta functions involving $E_{m,n}$.

Theorem 4.1. If $n$ is any positive real $P := \{G_{n/3}G_{3n}\}^3$ and $Q := E_{3,n}^3$, then

\[ Q + \frac{1}{Q} = 2\sqrt{2} \left\{ P - \frac{1}{P} \right\}. \quad (4.1) \]
Proof. Using the Lemma 2.1 with the definition of \( E_{m,n} \), we obtain

\[
E_{m,n} = \left\{ \frac{\beta (1 - \beta)}{\alpha (1 - \alpha)} \right\}^{1/12}.
\]  

(4.2)

Employing the above equation (4.2) and definition of class invariant (1.11), (1.12) in the Lemma 2.6 with \( m = 3 \), we obtain (4.1)

Corollary 4.2.

\[
E_{3,9} = \left\{ 1 + 2^{2/3} - 2^{4/3} \right\}^{1/3}.
\]  

(4.3)

Proof. Putting \( n = 9 \) in the above Theorem 4.1, we obtain

\[
E_{3,3} + E_{3,9} = 2 \sqrt{2} \left\{ G_3^3 G_{27}^3 - G_3^{-3} G_{27}^{-3} \right\}.
\]  

(4.4)

Solving the above equation (4.4) with from the table of Chapter 34 of Ramanujan’s notebooks [4, p.189,190] \( G_3 = 2^{1/12} \) and \( G_{27} = 2^{1/12} \left( \sqrt{2} - 1 \right)^{-1/3} \), we obtain (4.3).

Theorem 4.3. If \( n \) is any positive real \( P := \{G_n/5 G_{5n}\}^2 \) and \( Q := E_{5,n}^{5/2} \), then

\[
Q + \frac{1}{Q} = 2 \left\{ P - \frac{1}{P} \right\}.
\]  

(4.5)

Proof. Using the equation (4.2) and definition of class invariant (1.11), (1.12) in the Lemma 2.7 with \( m = 5 \), we obtain (4.5).

Theorem 4.4. If \( n \) is any positive real \( P := \{G_n/7 G_{7n}\}^3 \) and \( Q := E_{7,n}^{2} \), then

\[
Q + \frac{1}{Q} + 7 = 2 \sqrt{2} \left\{ P + \frac{1}{P} \right\}.
\]  

(4.6)

Proof. Using the equation (4.2) and definition of class invariant (1.11), (1.12) in the Lemma 2.8 with \( m = 7 \), we obtain (4.6).

Theorem 4.5.

\[
E_{3,n} = \frac{f(q) \psi(-q^3)}{q^{-1/6} f(q^3) \psi(-q)}; \quad q := e^{-\pi \sqrt{2}}
\]  

(4.7)

If

\[
P := \frac{\psi(-q)}{q^{1/4} \psi(-q^3)} \quad \text{and} \quad Q := \frac{f(q)}{q^{1/12} f(q^3)},
\]  

(4.8)

\[
E_{3,n}^6 = \frac{P^4 + 9}{P^4(1 + P^4)}, \quad \text{if} \quad P^4 \neq -1.
\]  

(4.9)

Proof. Employing the definition of \( E_{m,n} \) with \( m = 3 \), we get

\[
E_{3,n} = \frac{f(q) \psi(-q^3)}{q^{-1/6} f(q^3) \psi(-q)}.
\]  

(4.10)

Raising the power by 6 in the above equation (4.10) with the Lemma 2.3, we deduce that

\[
E_{3,n}^6 = \frac{f^6(q) \psi^6(-q^3)}{q^{-1} f^6(q^3) \psi^6(-q)},
\]  

(4.11)

\[
E_{3,n}^6 = \frac{P^2 \left\{ \frac{P^4 + 9}{1 + P^4} \right\}}{P^6}.
\]  

(4.12)

On simplifying the above equation (4.12), we obtain (4.9).
Corollary 4.6.

\[ E_{3,3} = \left\{ 2 - \sqrt{3} \right\}^{1/3}. \]  

(4.13)

Proof. Putting \( n = 3 \) in the equation (4.8) and from Ramanujan’s Notebooks [4, p. 327], we have

\[ \frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = \sqrt{3} - 9. \]  

(4.14)

Employing the equation (2.13) and (4.14), we obtain

\[ P := \frac{\psi(-e^{-\pi})}{\psi(-e^{-3\pi})} = \sqrt{9 + 6\sqrt{3}}. \]  

(4.15)

Substituting (4.15) in (4.9), we obtain the required result. \( \Box \)

Theorem 4.7.

\[ E_{5,n} = \frac{f(q)\psi(-q^n)}{q^{-1/3}f(q^n)\psi(-q)}; \quad q := e^{-\pi\sqrt{5}}. \]  

(4.16)

If

\[ P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \text{and} \quad Q := \frac{f(q)}{q^{1/6}f(q^5)}; \quad \text{then} \]

\[ E_{5,n}^3 = \frac{P^2 + 5}{P^2(P^2 + 1)} \quad \text{if} \quad P^2 \neq -1. \]  

(4.17)

(4.18)

Proof. Employing the definition of \( E_{m,n} \) with \( m = 5 \), we get

\[ E_{5,n} = \frac{f(q)\psi(-q^5)}{q^{-1/3}f(q^n)\psi(-q)}. \]  

(4.19)

Raising the power by 3 in the above equation (4.19) with the Lemma 2.4, we deduce that

\[ E_{5,n}^3 = \frac{f^3(q)\psi^3(-q^5)}{q^{-1/3}f^3(q^n)\psi^3(-q)}. \]  

(4.20)

\[ E_{5,n}^3 = \frac{P \left\{ 5 + P^2 \right\}}{P^3}; \quad P \neq -1. \]  

(4.21)

On simplifying the above equation (4.21), we obtain (4.18). \( \Box \)

Corollary 4.8.

\[ E_{5,5} = \left\{ 9 - 4\sqrt{5} \right\}^{2/3}. \]  

(4.22)

Proof. Putting \( n = 5 \) in the equation (4.17) and from Ramanujan’s Notebooks [4, p. 327], we have

\[ \frac{\varphi(e^{-\pi})}{\varphi(e^{-5\pi})} = \sqrt{5\sqrt{5} - 10}. \]  

(4.23)

Employing the equation (2.14) and (4.23), we obtain

\[ P := \frac{\psi(-e^{-\pi})}{\psi(-e^{-5\pi})} = \sqrt{5\sqrt{5} + 10}. \]  

(4.24)

Substituting (4.24) in (4.18), we obtain the required result. \( \Box \)
Theorem 4.9.
\[ E_{9,n} = \frac{f(q)\psi(-q^n)}{q^{-2/3}f(q^n)\psi(-q)}; \quad q := e^{-\pi \sqrt{n}}. \] (4.25)

If
\[ P := \frac{\psi(-q)}{q\psi(-q^n)} \quad \text{and} \quad Q := \frac{f(q)}{q^{1/3}f(q^n)}, \]
then
\[ E_{9,n}^3 = \left\{ \frac{P + 3}{P(P + 1)} \right\}^2, \quad \text{if} \quad P \neq -1. \] (4.27)

**Proof.** Employing the definition of \( E_{m,n} \) with \( m = 9 \), we get
\[ E_{9,n} = \frac{f(q)\psi(-q^n)}{q^{-2/3}f(q^n)\psi(-q)}. \] (4.28)

Raising the power by 3 in the above equation (4.28) with the Lemma 2.5, we deduce that
\[ E_{9,n}^3 = f^3(q)\psi^3(-q^n). \] (4.29)

On simplifying the above equation (4.30), we obtain (4.27).

Corollary 4.10.
\[ E_{9,9} = \left\{ \frac{33s^2 - (39 + \sqrt{3})s - 21\sqrt{3} + 6}{54 - 31\sqrt{3}} \right\}^{1/3}. \] (4.31)

where \( s = (2\sqrt{3} + 2)^{1/3} \)

**Proof.** Putting \( n = 9 \) in the equation (4.26) and from Ramanujan’s Notebooks [4, p. 327] we have,
\[ P := \frac{\varphi(e^{-\pi})}{\varphi(e^{-9\pi})} = \frac{3}{1 + \sqrt{2(\sqrt{3} + 1)}}. \] (4.32)

Employing the equation (2.15) and (4.32), we obtain
\[ P := \frac{\psi(-e^{-\pi})}{\psi(-e^{-9\pi})} = \frac{(s^2 + 2s + \sqrt{3} + 1)(3 + \sqrt{3})}{2}. \] (4.33)

Substituting (4.33) in (4.27), we obtain the required result.

Theorem 4.11.
\[ E_{m,n} = \left\{ \frac{G_{n/m}}{G_{mn}} \right\}^2. \] (4.34)

**Proof.** Employing the Lemma 2.1 in the definition of \( E_{m,n} \), we obtain
\[ E_{m,n} = \left\{ \frac{\beta(1 - \beta)}{\alpha(1 - \alpha)} \right\}^{1/12}. \] (4.35)

Using the equation (1.11) and (1.12), we get
\[ \frac{G_{nm}}{G_{n/m}} = \left\{ \frac{\alpha(1 - \alpha)}{\beta(1 - \beta)} \right\}^{1/24}. \] (4.36)

By observing the equations (4.35) and (4.36), we obtain (4.34).
Corollary 4.12.

\[ E_{n,n} = G^{-2}_n. \]  (4.37)

**Proof.** Setting \( m = n \) in the above Theorem 4.7 with the value \( G_1 = 1 \), we obtain required result. \( \square \)

Corollary 4.13.

\[\begin{align*}
(i) \quad & E_{2,2} = 2^{3/8}(1 + \sqrt{2})^{-1/2}, \\
(ii) \quad & E_{4,3} = \left\{ 2 - \sqrt{3} \right\}^{1/3}, \\
(iii) \quad & E_{5,5} = \frac{3 - \sqrt{5}}{2}, \\
(iv) \quad & E_{9,9} = \left\{ \frac{2(\sqrt{3} + 1)\sqrt[3]{1/3} + 1}{2(\sqrt{3} - 1)\sqrt[3]{1/3} - 1} \right\}^{-2/3}.
\end{align*}\]  (4.38) to (4.41)

**Proof.** For (i), we use the values of \( G_4 \) from [3, p.114, Theorem 6.2.2(ii)]. For (ii) – (iv), we use corresponding values of \( G_n \) from [2, p.189-193]. \( \square \)

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