Some Remarks on Multivalent Functions of Higher-order Derivatives

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ABSTRACT: In this paper we give necessary conditions for a suitably normalized multivalent function \( f(z) \) to be in the class \( G_{p,q}(\beta) \) of \( p \)-valently starlike functions of higher-order derivatives. Also we drive some properties of functions belonging to the class \( J_{p,q}(\alpha, \beta, f(z)) \) which consisting of multivalent \( \alpha \)-convex functions of higher-order derivatives in the unit disc.

Key Words: \( p \)-valent functions, Higher-order derivatives, \( \alpha \)-convex functions.

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1. Introduction

Let \( U = \{ z : |z| < 1 \} \) be the open unit disc of the complex plane \( \mathbb{C} \) and let \( A_p \) denote the class of analytic and \( p \)-valent functions in \( U \) of the form:

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \ldots\})
\]

(1.1)

Also let \( A_1 := A \). For two functions \( f, g \), we say that the function \( f \) is subordinate to \( g \) in \( U \), written as \( f(z) \prec g(z) \), (or simply \( f \prec g \)) if there exists a Schwarz function \( \omega \) analytic \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \), such that \( f(z) = g(\omega(z)) \). If the function \( g \) is univalent in \( U \), the subordination is equivalent to \( f(0) = g(0) \) and \( f(U) \subset g(U) \) (see [8]). For \( 0 \leq \beta < p - q, p > q, p \in \mathbb{N} \) and \( q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), we say that \( f(z) \in A_p \) is in the class \( S^*_{p,q}(\beta) \) if it satisfies the following inequality

\[
\Re \left\{ \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right\} > \beta \quad (z \in U).
\]

(1.2)

Also, for \( 0 \leq \beta < p - q, p > q, p \in \mathbb{N} \) and \( q \in \mathbb{N}_0 \), we say that \( f(z) \in A_p \) is in the class \( K_{p,q}(\beta) \) if it satisfies the following inequality

\[
\Re \left\{ 1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} \right\} > \beta \quad (z \in U).
\]

(1.3)

It follows from (1.2) and (1.3) that

\[
f(z) \in K_{p,q}(\beta) \iff F(z) \in S^*_{p,q}(\beta),
\]

where \( F \in A_p \), such that \( F^{(q)}(z) = \frac{zf^{(q+1)}(z)}{p-q} \) \((z \in U)\). The classes \( S^*_{p,q}(\beta) \) and \( K_{p,q}(\beta) \) were introduced and studied by Aouf [2,3,4]. We note that \( S^*_{p,0}(\beta) \equiv S^*_{p}(\beta) \) and \( K_{p,0}(\beta) \equiv K_{p}(\beta) \) are, respectively, the class of \( p \)-valently starlike functions of order \( \beta \) and the class of \( p \)-valently convex functions of order \( \beta \) \((0 \leq \beta < p)\) see Owa [12] and Aouf [1].

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Let $G_{p,q}(\beta)$ denote the subclass of $A_p$ consisting of functions $f(z)$ which satisfy

\[
\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} < (p - q) + (p - q - \beta)z \quad (0 \leq \beta < p - q, p > q).
\]  \hspace{1cm} (1.4)

It is clear that (1.4) is equivalent to

\[
\left| \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p - q) \right| < (p - q - \beta) \quad (z \in \mathbb{U}).
\]  \hspace{1cm} (1.5)

Therefore $G_{p,q}(\beta)$ is a subclass of the class $S^*_{p,q}(\beta)$.

A function $f(z) \in A_p$ is said to be $p$–valently $\alpha$–convex functions of higher order derivatives of order $\beta$ if it satisfies

\[
\Re \left\{ (1 - \alpha) \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + \alpha \left( 1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} \right) \right\} > \beta
\]  \hspace{1cm} (1.6)

for some $\alpha (\alpha \geq 0), \beta (0 \leq \beta < \delta(p,q))$ and for all $(z \in \mathbb{U})$, where

\[
\delta(p,q) = \frac{p!}{(p-q)!} \quad (p > q).
\]

Denoting by $J_{p,q}(\alpha, \beta, f(z))$ the subclass of $A_p$ consisting of all such functions. We note that $J_{p,q}(0, \beta, f(z)) \cong S^*_{p,q}(\beta)$ and $J_{p,q}(1, \beta, f(z)) \cong K_{p,q}(\beta)$. Also we note that $J_{p,1-p}(\alpha, 0, f(z)) \cong A(p, \alpha)$ ($p \in \mathbb{N}, \alpha \geq 1$) was introduced and studied by Nunokawa [9], Saitoh et al. [14] and Nishimoto and Owa [11] and $J_{p,0}(\alpha, \beta, f(z)) \cong M(p, 1, \alpha, \beta)$ was introduced and studied by Owa [13].

2. Main Results

In order to prove our results we need the following lemmas.

**Lemma 2.1.** [6] Let $\omega(z)$ be regular in $\mathbb{U}$ with $\omega(0) = 0$. Then if \(|\omega(z)| \) attains its maximum value on the circle \(|z| = r \) at a point $z_0 \in \mathbb{U}$, we have $z_0 \omega(z_0) = m \omega(z_0)$, where $m \geq 1$.

**Lemma 2.2.** [7] Let $\phi(z)$ be a complex valued function

\[
\phi : D \rightarrow C, D \subset C \times C \quad (C \text{ is the complex plane}).
\]

and let $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose that the function $\phi(u, v)$ satisfies

(i) $\phi(u, v)$ is continuous in $D$;
(ii) $(1, 0) \in D$ and $\Re\{\phi(1, 0)\} > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{(1 + u_2^2)}{2}, \Re\{\phi(iu_2, v_1)\} \leq 0$.

Let $p(z) = 1 + p_1z + p_2z^2 + ...$ be regular in the unit disc $\mathbb{U}$, such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

\[
\Re\{\phi(p(z), zp'(z))\} > 0 \quad (z \in \mathbb{U}),
\]

then $\Re\{p(z)\} > 0 \quad (z \in \mathbb{U})$.

**Theorem 2.3.** If $f(z) \in A_p$ satisfies

\[
\left| \lambda \left( \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p - q) \right) + (1 - \lambda) \left( \frac{z^2f^{(q+2)}(z)}{f^{(q)}(z)} - (p - q)(p - q - 1) \right) \right| < (p - q - \beta)\lambda + (1 - \lambda)(p - q + \beta) \quad (z \in \mathbb{U}),
\]  \hspace{1cm} (2.1)

for some $(0 \leq \beta < p - q, p > q, p \in \mathbb{N}, q \in \mathbb{N}_0$ and $0 \leq \lambda < 1$), then $f(z) \in G_{p,q}(\beta)$. 
**Proof:** Define the function $\omega(z)$ by

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = (p - q) + (p - q - \beta)\omega(z). \quad (2.2)$$

Then, $\omega(z)$ is regular in $U$ and $\omega(0) = 0$. Differentiating (2.2) logarithmically with respect to $z$, we obtain

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - (p - q) = (p - q - \beta)\omega(z) + \frac{(p - q - \beta)z\omega'(z)}{(p - q) + (p - q - \beta)\omega(z)}. \quad (2.3)$$

From (2.2) and (2.3), we have

$$\frac{z^2f^{(q+2)}(z)}{f^{(q)}(z)} - (p - q)(p - q - 1) = (p - q - \beta)\omega(z)[2(p - q) - 1 + (p - q - \beta)\omega(z) + \frac{z\omega'(z)}{\omega(z)}]. \quad (2.4)$$

From (2.2) and (2.4), we have

$$\lambda \left( \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p - q) \right) + (1 - \lambda) \left( \frac{z^2f^{(q+2)}(z)}{f^{(q)}(z)} - (p - q)(p - q - 1) \right)
= (p - q - \beta)\omega(z)
\times \left\{ \lambda + (1 - \lambda)[2(p - q) - 1 + (p - q - \beta)\omega(z) + \frac{z\omega'(z)}{\omega(z)}] \right\}. \quad (2.5)$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1.$$  

Then by using Lemma 2.1, and letting $\omega(z_0) = e^{i\theta}$, we get

$$\left| \lambda \left( \frac{zf^{(q+1)}(z_0)}{f^{(q)}(z_0)} - (p - q) \right) + (1 - \lambda) \left( \frac{z^2f^{(q+2)}(z_0)}{f^{(q)}(z_0)} - (p - q)(p - q - 1) \right) \right|
= |(p - q - \beta)\omega(z_0)|
\left\{ \lambda + (1 - \lambda)[2(p - q) - 1 + \frac{z\omega'(z_0)}{\omega(z_0)}] + (1 - \lambda)(p - q - \beta)\omega(z_0) \right\}
= (p - q - \beta) |\lambda + (1 - \lambda)[2(p - q) - 1 + k] + (1 - \lambda)(p - q - \beta)e^{i\theta}| \geq (p - q - \beta)[|\lambda + (1 - \lambda)(p - q + \beta)|].$$

This contradicts the condition (2.1). Therefore $|\omega(z)| < 1$ for all $z \in U$. This implies that

$$\left| \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p - q) \right| < p - q - \beta \quad (z \in U),$$

that is $f(z) \in G_{p,q}(\beta)$. This completes the proof of Theorem 2.3.

Taking $q = 0$ in Theorem 2.3, we have
Corollary 2.4. If \( f(z) \in A_p \) satisfies
\[
\left| \lambda \left( \frac{zf'(z)}{f(z)} - 1 \right) + (1 - \lambda) \left( 1 + \frac{z^2f''(z)}{f(z)} - p(p-1) \right) \right| < (p-\beta)[\lambda + (1-\lambda)(p+\beta)] \quad (z \in U),
\]
then \( f(z) \in G_p(\beta) := \left\{ f(z) \in A_p : \left| \frac{zf'(z)}{f(z)} - p \right| < p - \beta \quad (z \in U) \right\}. \)

Putting \( p = 1 \) in Corollary 2.4, we have

Corollary 2.5. If \( f(z) \in A \) satisfies
\[
\left| \lambda \left( \frac{zf'(z)}{f(z)} - 1 \right) + (1 - \lambda) \frac{z^2f''(z)}{f(z)} \right| < (1 - \beta)[\lambda + (1-\lambda)(1+\beta)] \quad (z \in U),
\]
then \( f(z) \in G(\beta) := \left\{ f(z) \in A : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \beta \quad (z \in U) \right\}. \)

This corollary is an improvement of the results obtained by Fukui [5, Theorem 1] and Nunokawa and Hoshino [10, Theorem 1].

Putting \( \lambda = 0 \) in Theorem 2.3, we have

Corollary 2.6. If \( f(z) \in A_p \) satisfies
\[
\left| \frac{z^2f(q+2)(z)}{f(q)(z)} - (p-q)(p-q-1) \right| < (p-\beta)(p-q+\beta) \quad (z \in U),
\]
for some \( 0 \leq \beta < p-q, p > q, p \in \mathbb{N} \) and \( q \in \mathbb{N}_0 \), then \( f(z) \in G_{p,q}(\beta) \).

Putting \( q = 0 \) in Corollary 2.6, we obtain the following corollary

Corollary 2.7. If \( f(z) \in A_p \) satisfies
\[
\left| \frac{z^2f''(z)}{f(z)} - p(p-1) \right| < (p^2 - \beta^2) \quad (z \in U),
\]
for some \( 0 \leq \beta < p \), then \( f(z) \in G_p(\beta) \).

Remark 2.8. Our result in Corollary 2.7 when \( p = 1 \) is an improvement of the results obtained by Fukui [5, Corollary 1] and Nunokawa and Hoshino [10, Corollary 2].

Putting \( \lambda = \frac{1}{2} \) in Corollary 2.4, we obtain

Corollary 2.9. If \( f(z) \in A_p \) satisfies
\[
\left| \frac{zf'(z) + z^2f''(z)}{f(z)} - p \right| < (p - \beta)(p + 1 + \beta) \quad (z \in U),
\]
for some \( 0 \leq \beta < p \), then \( f(z) \in G_p(\beta) \).

Remark 2.10. Our result in Corollary 2.9 when \( p = 1 \) is an improvement of the results obtained by Fukui [5, Corollary 2] and Nunokawa and Hoshino [10, Corollary 3].
**Theorem 2.11.** Let the function $f(z)$ defined by (1.1) belongs to the class $J_{p,q}(\alpha, f(z))$ with $p > q, p \in \mathbb{N}, q \in \mathbb{N}_0$ and $\alpha \geq 1$, then

$$\Re \left\{ \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right\} > \beta = -\alpha + \frac{\sqrt{\alpha(\alpha + 8\delta(p,q))}}{4} \quad (2.6)$$

**Proof:** Define the function $g(z)$ by

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = \beta + [\delta(p,q) - \beta]g(z), \quad 0 \leq \beta < \delta(p,q) \quad (2.7)$$

for $f(z) \in J_{p,q}(\alpha, f(z))$, where

$$\beta = -\alpha + \frac{\sqrt{\alpha(\alpha + 8\delta(p,q))}}{4} \quad (2.8)$$

It follows from (2.7) that $g(z)$ is regular in $\mathbb{U}$ and that $g(z) = 1 + g_1z + g_2z^2 + \ldots$. Differentiating (2.7) logarithmically with respect to $z$, we obtain

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} = \beta + [\delta(p,q) - \beta]g(z) + \frac{[\delta(p,q) - \beta]g'(z)}{\beta + [\delta(p,q) - \beta]g(z)}. \quad (2.9)$$

From (2.7) and (2.9), we have

$$\Re \left\{ (1 - \alpha) \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + \alpha \left( 1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} \right) \right\} = \Re \left\{ \beta + [\delta(p,q) - \beta]g(z) + \frac{\alpha[\delta(p,q) - \beta]g'(z)}{\beta + [\delta(p,q) - \beta]g(z)} \right\} > 0. \quad (2.10)$$

Letting $u = u_1 + iu_2, v = v_1 + iv_2$ and

$$\phi(u, v) = \beta + [\delta(p,q) - \beta]u + \frac{\alpha[\delta(p,q) - \beta]u}{\beta + [\delta(p,q) - \beta]u}, \quad (2.11)$$

we know that

(i) $\phi(u, v)$ is continuous in $D = \left(C - \frac{\beta}{\alpha - \delta(p,q)}\right) \times C$;

(ii) $(1,0) \in D$ and $\Re\{\phi(1,0)\} = \delta(p,q) > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{(1 + u_2^2)}{2}$,

$$\Re\{\phi(iu_2, v_1)\} = \beta + \frac{\alpha[\delta(p,q) - \beta]v_1}{\beta^2 + [\delta(p,q) - \beta]^2u_2^2} \leq \beta - \frac{\alpha[\delta(p,q) - \beta](1 + u_2^2)}{2(\beta^2 + [\delta(p,q) - \beta]^2u_2^2)} \leq 0.$$

Therefore, the function $\phi(u, v)$ defined by (2.11) satisfies the conditions of Lemma 2.2. It follows from this fact that $\Re\{g(z)\} > 0$, that is that

$$\Re \left\{ \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right\} > \beta = -\alpha + \frac{\sqrt{\alpha(\alpha + 8\delta(p,q))}}{4}.$$

This completes the proof of Theorem 2.11. □

Putting $q = 1 - 1(p \in \mathbb{N})$ in Theorem 2.11, we obtain the following corollary.
Corollary 2.12. Let the function $f(z)$ defined by (1.1) belongs to the class 
$$J_{p,1-p}(\alpha, f(z)) = J_p(\alpha, f(z))$$
with $p \in \mathbb{N}$ and $\alpha \geq 1$, then
$$\Re \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > \beta = \frac{-\alpha + \sqrt{\alpha(\alpha + 8p!)}}{4}.$$

From the definition of the class $J_p(\alpha, f(z))$ and Theorem 2.11, we have

Corollary 2.13. Let the function $f(z)$ defined by (1.1) belongs to the class $J_{p,q}(\alpha, f(z))$ with $p > q, p \in \mathbb{N}, q \in \mathbb{N}_0$ and $\alpha \geq 1$, then
$$\Re \left\{ 1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} \right\} > \beta = \frac{(\alpha - 1)(-\alpha + \sqrt{\alpha(\alpha + 8\delta(p,q))})}{4\alpha}.$$

Putting $q = p - 1(p \in \mathbb{N})$ in Corollary 2.13, we obtain the following corollary

Corollary 2.14. Let the function $f(z)$ defined by (1.1) belongs to the class $J_p(\alpha, f(z))$ with $p \in \mathbb{N}$ and $\alpha \geq 1$, then
$$\Re \left\{ 1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right\} > \beta = \frac{(\alpha - 1) + \sqrt{\alpha(\alpha + 8p!)}}{4\alpha}.$$

Remark 2.15. Our result in Corollary 2.14 is an improvement of the result obtained by Saitoh et al. [14, Corollary 1].

Putting $\alpha = 1$ in Corollary 2.12, we have

Corollary 2.16. Let the function $f(z)$ defined by (1.1) be in the class $K_p$, then
$$\Re \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > \beta = \frac{-1 + \sqrt{1 + 8p!}}{4}\frac{1}{2}.$$

Remark 2.17. Putting $p = 1$ in Corollary 2.16, then if the function $f(z) \in A$ is convex in $\mathbb{U}$, then $f(z)$ is starlike of order $\frac{1}{2}$ in $\mathbb{U}$ (see also Saitoh et al. [14, Corollary 2]).

References


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