On Generalized Weakly Symmetric Kenmotsu Manifolds

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ABSTRACT: This paper aims to introduce the notions of a generalized weakly symmetric Kenmotsu manifolds and a generalized weakly Ricci-symmetric Kenmotsu manifolds. The existence of a generalized weakly symmetric Kenmotsu manifold is ensured by a non-trivial example.

Key Words: Generalized weakly symmetric Kenmotsu manifolds, Generalized weakly Ricci-symmetric Kenmotsu manifolds.

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1. Introduction

The notion of weakly symmetric Riemannian manifold have been introduced by Tamássy and Binh [12]. Thereafter, a lot of research has been carried out in this topic. For details, we refer to [21], [19], [5], [6], [15], [16], [1], [20], [7] and the references there in. In the spirit of [12], a Kenmotsu manifold \((M^n, g) (n \geq 2)\), is said to be a weakly symmetric manifold, if its curvature tensor \(\bar{R}\) of type \((0, 4)\) is not identically zero and admits the identity

\[
\]

where \(A_1, B_1 & D_1\) are non-zero 1-forms defined by \(A_1(X) = g(X, \sigma_1)\), \(B_1(X) = g(X, g_1)\) and \(D_1(X) = g(X, \pi_1)\), for all \(X\) and \(\bar{R}(Y, U, V, W) = g(R(Y, U)V, W)\), \(\nabla\) being the operator of the covariant differentiation with respect to the metric tensor \(g\). An \(n\)-dimensional Kenmotsu manifold of this kind is denoted by \((WS)_{n}\)-manifold.
Keeping in tune with Dubey [17], we shall call a Kenmotsu manifold of dimension $n$, a generalized weakly symmetric (which is abbreviated hereafter as $(GW S)_n$-manifold) if it admits the equation

$$
$$

(1.2)

where

$$
\tilde{G}(Y, U, V, W) = [g(U, V)g(Y, W) - g(Y, V)g(U, W)]
$$

(1.3)

and $A_i$, $B_i$ and $D_i$ are non-zero 1-forms defined by $A_i(X) = g(X, \sigma_i)$, $B_i(X) = g(X, \xi_i)$, and $D_i(X) = g(X, \pi_i)$, for $i = 1, 2$. The beauty of such $(GW S)_n$-manifold is that it has the flavour of

(i) locally symmetric space [4] (for $A_1 = B_1 = D_1 = 0$),

(ii) recurrent space [2] (for $A_1 \neq 0$, $A_2 = A_1 = D_1 = 0$),

(iii) generalized recurrent space [17] (for $A_1 \neq 0$, $B_1 = D_1 = 0$),

(iv) pseudo symmetric space [13] (for $A_1 = B_1 = D_1 = H_1 \neq 0$, $A_2 = B_2 = D_2 = 0$),

(v) generalized pseudo symmetric space [9] (for $A_1 = B_1 = D_1 = H_1 \neq 0$),

(vi) semi-pseudo symmetric space [14] (for $A_i = B_2 = D_2 = 0$, $B_1 = D_1 \neq 0$),

(vii) generalized semi-pseudo symmetric space [8] (for $A_1 = 0$, $B_1 = D_1 \neq 0$),

(viii) almost pseudo symmetric space [13] (for $A_1 = H_1 + K_1$, $B_1 = D_1 = H_1 \neq 0$ and $A_2 = B_2 = D_2 = 0$),

(ix) almost generalized pseudo symmetric space [10] (or $A_1 = H_1 + K_1$, $B_1 = D_1 = H_1 \neq 0$),

(x) weakly symmetric space [12] (for $A_1$, $B_1$, $D_1 \neq 0$, $A_2 = B_2 = D_2 = 0$).

Our work is structured as follows. Section 2 is concerned with Kenmotsu manifolds and some known results. In section 3, we have investigated a generalized weakly symmetric Kenmotsu manifold and it is observed that such a space is an $\eta$-Einstein manifold provided $D_1(\xi) \neq -1$. We also tabled different type of curvature restrictions for which Kenmotsu manifolds are sometimes Einstein and some other time remain $\eta$-Einstein. Section 4, is concerned with a generalized weakly Ricci-symmetric Kenmotsu manifold which is also found to be $\eta$-Einstein space. Finally, we have constructed an example of a generalized weakly symmetric Kenmotsu manifold.

2. Kenmotsu manifolds and some known results

Let $M$ be a $n$-dimensional connected differentiable manifold of class $C^\infty$-covered by a system of coordinate neighborhoods $(U, x^i)$ in which there are given a tensor field $\phi$ of type $(1, 1)$, a cotriavariant vector field $\xi$ and a 1-form $\eta$ such that
\[ \phi^2 X = -X + \eta(X)\xi, \]  
\[ \eta(\xi) = 1, \quad \phi \cdot \xi = 0, \quad \eta(\phi X) = 0, \]

for any vector field \( X \) on \( M \). Then the structure \( (\phi, \xi, \eta) \) is called contact structure and the manifold \( M^n \) equipped with such structure is said to be an almost contact manifold, if there is given a Riemannian compatible metric \( g \) such that

\[ g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X), \]
\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \]

for all vector fields \( X \) and \( Y \), then we say \( M \) is an almost contact metric manifold.

An almost contact metric manifold \( M \) is called a Kenmotsu manifold if it satisfies

\[ R(X, Y)\xi = \eta(X) Y - \eta(Y)X, \]
\[ S(X, \xi) = -(n-1)\eta(X), \]
\[ R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \]
\[ R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi \]

for any vector fields \( X, Y, Z \) where \( R \) is the Riemannian curvature tensor of the manifold.

3. Generalized weakly symmetric Kenmotsu manifold

In this section, we consider a generalized weakly symmetric Kenmotsu manifold \((M^n, g)(n > 2)\). Now, contracting \( Y \) over \( W \) in both sides of (1.2), we get

\[ (\nabla_X S)(U, V) = A_1(X)S(U, V) + B_1(U)S(X, V) + B_1(R(X, U)V) + D_1(R(X, V)U) + D_1(V)S(U, X) + (n-1)[A_2(X)g(U, V) + B_2(U)g(X, V) + D_2(V)g(U, X)] + B_2(G(X, U)V) + D_2(G(X, V)U). \]
As a consequence of (2.8), (2.9) and (2.10) the above equation yields

\[
(\nabla_X S)(U, \xi) = -(n - 1)A_1(X)\eta(U) - (n - 2)B_1(U)\eta(X) \\
+ D_1(\xi)S(U, X) - \eta(U)B_1(X) - \eta(U)D_1(X) + g(X, U)D_1(\xi) \\
+ (n - 1)[A_2(X)\eta(U) + B_2(U)\eta(X) + D_2(\xi)g(U, X)] \\
+ \eta(U)B_2(X) - \eta(X)B_2(U) + \eta(U)D_2(X) - g(U, X)D_2(\xi)
\]

for \( V = \xi \). Again, replacing \( V \) by \( \xi \), in the following identity

\[
(\nabla_X S)(U, V) = \nabla_X S(U, V) - S(\nabla_X U, V) - S(U, \nabla_X V)
\]

(3.1)

and then making use of (2.1), (2.6), (2.9), we find

\[
(\nabla_X S)(U, \xi) = -(n - 1)g(X, U) - S(U, X).
\]

(3.2)

Now, using (3.2) in (3.1), we have

\[
-(n - 1)g(X, U) - S(U, X) \quad (3.3)
\]

\[
= -(n - 1)A_1(X)\eta(U) - (n - 2)B_1(U)\eta(X) \\
+ D_1(\xi)S(U, X) - \eta(U)B_1(X) + g(X, U)D_1(\xi) - \eta(U)D_1(X) \\
+ (n - 1)[A_2(X)\eta(U) + B_2(U)\eta(X) + D_2(\xi)g(U, X)] \\
+ \eta(U)B_2(X) - \eta(X)B_2(U) + \eta(U)D_2(X) - g(U, X)D_2(\xi)
\]

which leaves

\[
[A_1(\xi) + B_1(\xi) + D_1(\xi)] = [A_2(\xi) + B_2(\xi) + D_2(\xi)]
\]

(3.4)

for \( X = U = \xi \). In particular, if \( A_2(\xi) = B_2(\xi) = D_2(\xi) = 0 \), (3.4) turns into

\[
A_1(\xi) + B_1(\xi) + D_1(\xi) = 0.
\]

(3.5)

This leads to the following

**Theorem 3.1.** In a generalized weakly symmetric Kenmotsu manifold \((M^n, g)\) \((n > 2)\), the relation (3.4) hold good.
Putting $X = \xi$ in (3.6) and using (2.1), (2.2), (2.9), we obtain

$$
(n-1)[A_1(\xi) + B_1(\xi)]\eta(V) + (n-2)D_1(V) + \eta(V)D_1(\xi) = (n-1)\{A_2(\xi) + B_2(\xi)\} \eta(V) + D_2(V) + \eta(V)D_2(\xi) - D_2(V).
$$

Replacing $V$ by $X$ in the above equation and using (3.4), we get

$$
D_1(X) - D_1(\xi)\eta(X) = D_2(X) - D_2(\xi)\eta(X).
$$

Moreover, in view of (3.4), (3.7) and (3.9), we get

$$
B_1(X) - B_1(\xi)\eta(X) = B_2(X) - B_2(\xi)\eta(X).
$$

Subtracting (3.9), (3.10) from (3.7), we get

$$
A_1(X) + \{B_1(\xi) + D_1(\xi)\} \eta(X) = A_2(X) + \{B_2(\xi) + D_2(\xi)\} \eta(X).
$$

Again, adding (3.9), (3.10) and (3.11), we get

$$
A_1(X) + B_1(X) + D_1(X) = [A_2(X) + B_2(X) + D_2(X)].
$$

Next, for the choice of $A_2 = B_2 = C_2 = D_2 = 0$, the relation (3.12) yields

$$
A_1(X) + B_1(X) + D_1(X) = 0.
$$

This motivates us to state the followings

**Theorem 3.2.** In a generalized weakly symmetric Kenmotsu manifold $(M^n,g)(n > 2)$, the sum of the associated 1-forms is given by (3.12).

**Theorem 3.3.** There does not exist a Kenmotsu manifold which is

(i) recurrent,

(ii) generalized recurrent provided the 1-forms associated to the vector fields are collinear,

(iii) pseudo symmetric,

(iv) generalized semi-pseudo symmetric provided the 1-forms associated to the vector fields are collinear.

Again from (3.3), putting $X = \xi$, we have

$$
(n-1)[-\{A_1(\xi) - A_2(\xi)\} - \{D_1(\xi) - D_2(\xi)\}]\eta(U) = \{B_1(\xi) - B_2(\xi)\} \eta(U) + (n-2)\{B_1(U) - B_2(U)\}.
$$

Using (3.4), above equation becomes

$$
\{B_1(\xi) - B_2(\xi)\} \eta(U) = B_1(U) - B_2(U).
$$

Setting $U = \xi$, we have

$$
-(n-1)[A_1(X) - A_2(X)] - \{B_1(X) - B_2(X)\} - \{D_1(X) - D_2(X)\} = -(n-2)\{A_1(\xi) - A_2(\xi)\}.
$$
Using (3.4) in (3.16), we obtain

$$\{A_1(\xi) - A_2(\xi)\} \eta(X) = A_1(X) - A_2(X). \quad (3.17)$$

Again from (3.3), we have

$$S(U, X) = \left[\frac{(n-1)(A_1(X) - A_2(X)) + B_1(X) - B_2(X) + [D_1(X) - D_2(X)]}{[1 + D_1(\xi)]} \eta(U) \right.$$

$$\left. - \frac{[n-1][1 + D_2(\xi)] + D_1(\xi) - D_2(\xi)}{[1 + D_1(\xi)]} g(X, U) \right]$$

$$\left. - \frac{(n-2)[B_1(U) - B_2(U)]}{[1 + D_1(\xi)]} \eta(X) \right]. \quad (3.18)$$

In view of (3.15), (3.17) and (3.18), we have

$$S(U, X) = - \left[\frac{(n-1)([1 + D_2(\xi)] + D_1(\xi) - D_2(\xi))}{[1 + D_1(\xi)]} g(X, U) \right.$$  

$$\left. + \frac{(n-2)[A_1(\xi) - A_2(\xi) - (B_1(\xi) - B_2(\xi))]}{[1 + D_1(\xi)]} \eta(U) \eta(X) \right]. \quad (3.19)$$

This leads to the followings

**Theorem 3.4.** A generalized weakly symmetric Kenmotsu manifold is an $\eta$-Einstein space provided $D_1(\xi) \neq -1$.

**Theorem 3.5.** In a Kenmotsu manifold the following table hold good

<table>
<thead>
<tr>
<th>Type of curvature restriction</th>
<th>Nature of the space corresponding to curvature restriction</th>
</tr>
</thead>
<tbody>
<tr>
<td>locally symmetric space</td>
<td>Einstein space</td>
</tr>
<tr>
<td>locally recurrent space</td>
<td>$\eta$-Einstein space</td>
</tr>
<tr>
<td>generalized recurrent space</td>
<td>$\eta$-Einstein space</td>
</tr>
<tr>
<td>pseudo symmetric space</td>
<td>$\eta$-Einstein space</td>
</tr>
<tr>
<td>generalized pseudo symmetric space</td>
<td>$\eta$-Einstein space</td>
</tr>
<tr>
<td>semi-pseudo symmetric space</td>
<td>$\eta$-Einstein space</td>
</tr>
<tr>
<td>generalized semi-pseudo symmetric space</td>
<td>$\eta$-Einstein space</td>
</tr>
<tr>
<td>almost pseudo symmetric space</td>
<td>$\eta$-Einstein space</td>
</tr>
<tr>
<td>almost generalized pseudo symmetric space</td>
<td>$\eta$-Einstein space</td>
</tr>
<tr>
<td>weakly symmetric space</td>
<td>$\eta$-Einstein space</td>
</tr>
</tbody>
</table>
4. Generalized weakly Ricci-symmetric Kenmotsu manifold

A Kenmotsu manifold \((M^n, g)(n > 2)\), is said to be a generalized weakly Ricci-symmetric if there exist 1-forms \(\bar{A}_1, \bar{B}_1\) and \(\bar{D}_1\) which satisfy the condition

\[
(\nabla_X S)(U, V) = \bar{A}_1(X)S(U, V) + \bar{B}_1(U)S(X, V) + \bar{D}_1(V)S(U, X) + \bar{A}_2(X)g(U, V) + \bar{B}_2(U)g(X, V) + \bar{D}_2(V)g(U, X).
\]  (4.1)

Putting \(V = \xi\) in (4.1), we obtain

\[
(\nabla_X S)(U, \xi) = (n - 2)[\bar{A}_1(X)\eta(U) + \bar{B}_1(U)\eta(X)] + \bar{D}_1(\xi)S(U, X) + \bar{A}_2(X)\eta(U) + \bar{B}_2(U)\eta(X) + \bar{D}_2(\xi)g(U, X).
\]  (4.2)

In view of (3.2), the relation (4.2) becomes

\[
-(n - 1)g(X, U) - S(U, X) = -(n - 1)[\bar{A}_1(X) + \bar{B}_1(X)]\eta(U) + \bar{B}_1(U)\eta(X) + \bar{D}_1(\xi)S(U, X) + \bar{A}_2(X)\eta(U) + \bar{B}_2(U)\eta(X) + \bar{D}_2(\xi)g(U, X).
\]  (4.3)

Setting \(X = U = \xi\) in (4.3) and using (2.1), (2.2) and (2.9), we get

\[
(n - 1)[\bar{A}_1(\xi) + \bar{B}_1(\xi) + \bar{D}_1(\xi)] = [\bar{A}_2(\xi) + \bar{B}_2(\xi) + \bar{D}_2(\xi)].
\]  (4.4)

Again, putting \(X = \xi\) in (4.3), we get

\[
(n - 1)[\bar{A}_1(\xi) + \bar{B}_1(\xi) + \bar{D}_1(\xi)]\eta(U) + \bar{B}_1(U)] = [\bar{A}_2(\xi) + \bar{B}_2(\xi) + \bar{D}_2(\xi)]\eta(U) + \bar{B}_2(U).
\]  (4.5)

Setting \(U = \xi\) in (4.3) and then using (2.1), (2.2) and (2.9), we obtain

\[
(n - 1)[\bar{A}_1(X) + \bar{B}_1(X)] + \{\bar{B}_1(\xi) + \bar{D}_1(\xi)\}g(X)\eta(X) = \bar{A}_2(X) + \bar{B}_2(\xi)\eta(X) + \bar{D}_2(\xi)\eta(X).
\]  (4.6)

Replacing \(U\) by \(X\) in (4.5) and then adding the resultant with (4.6), we have

\[
(n - 1)[\bar{A}_1(X) + \bar{B}_1(X)] = \bar{A}_2(X) + \bar{B}_2(\xi)\eta(X).
\]  (4.7)

By virtue of (4.4), the above equation becomes

\[
(n - 1)[\bar{A}_1(X) + \bar{B}_1(\xi)] + (n - 1)\bar{D}_1(\xi)\eta(X) = \bar{A}_2(X) + \bar{B}_2(\xi)\eta(X).
\]  (4.8)

Next, putting \(X = U = \xi\) in (4.1), we get

\[
(n - 1)[\bar{A}_1(\xi) + \bar{B}_1(\xi)]g(V) + (n - 1)\bar{D}_1(V) = [\bar{A}_2(\xi) + \bar{B}_2(\xi)]g(V) + \bar{D}_2(V).
\]  (4.9)
Replacing $V$ by $X$ in (4.9) and adding with (4.8), we obtain

$$(n-1)[\bar{A}_1(X) + \bar{B}_1(X) + \bar{D}_1(X)]$$

(4.10)

$$(n-1)[\bar{A}_1(\xi) + \bar{B}_1(\xi) + \bar{D}_1(\xi)]\eta(V)$$

$$= [\bar{A}_2(X) + \bar{B}_2(X) + \bar{D}_1(X)] + [\bar{A}_2(\xi) + \bar{B}_2(\xi) + \bar{D}_2(\xi)]\eta(V).$$

By virtue of (4.4), the above equation becomes

$$(n-1)[\bar{A}_1(X) + \bar{B}_1(X) + \bar{D}_1(X)] = [\bar{A}_2(X) + \bar{B}_2(X) + \bar{D}_1(X)].$$

(4.11)

This leads to the followings

**Theorem 4.1.** In a generalized weakly Ricci symmetric Kenmotsu manifold $(M^n, g)$ $(n > 2)$, the sum of the associated 1-forms are related by (4.11).

Again from (4.3), we have

$$S(U, X) = \frac{1}{-[1 + D_1(\xi)]}[\bar{D}_2(\xi) + (n-1)g(X, U)]$$

(4.12)

$$+ \frac{1}{[1 + D_1(\xi)]}[(n-1)\{\bar{A}_1(X) + \bar{B}_1(X)\} - \bar{A}_2(X)]\eta(U)$$

$$+ \frac{1}{[1 + D_1(\xi)]}[(n-1)\bar{B}_1(U) - \bar{B}_2(U)]\eta(X).$$

From (4.6), we have

$$(n-1)[\{\bar{A}_1(X) + \bar{B}_1(X)\} - \bar{A}_2(X)]$$

(4.13)

$$= [-\{(n-1)\{\bar{B}_1(\xi) + \bar{D}_1(\xi)\} + \bar{B}_2(\xi) + \bar{D}_2(\xi)\]}\eta(X).$$

Using (4.4) in (4.5), we have

$$(n-1)\bar{B}_1(U) - \bar{B}_2(U) = -(n-1)\bar{B}_1(\xi) + \bar{B}_2(\xi).$$

(4.14)

In view of (4.12), (4.13) and (4.14), we have

$$S(U, X)$$

(4.15)

$$= \frac{[\bar{D}_2(\xi) + (n-1)]}{-[1 + D_1(\xi)]}g(X, U)$$

$$+ \frac{[-2\{(n-1)\bar{B}_1(\xi) - \bar{B}_2(\xi)\} - \{(n-1)\bar{D}_1(\xi) - \bar{D}_2(\xi)\}]}{[1 + D_1(\xi)]}\eta(U)\eta(X).$$

This leads to the followings

**Theorem 4.2.** A generalized weakly Ricci symmetric Kenmotsu manifold is an $\eta$-Einstein space provided $D_1(\xi) \neq -1$.

**Theorem 4.3.** In a Kenmotsu manifold the following table hold good
5. Example of an (GWS)$_3$ Kenmotsu manifold

(see [18], page 21-22) Let $M^3(\phi, \xi, \eta, g)$ be a Kenmotsu manifold $(M^3, g)$ with a $\phi$-basis

$$e_1 = e^{-\phi} \frac{\partial}{\partial x}, \quad e_2 = e^{-\phi} \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$  

Then from Koszul’s formula for Riemannian metric $g$, we can obtain the Levi-Civita connection as follows

$$\nabla_{e_1} e_3 = e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = -e_3,$$
$$\nabla_{e_2} e_3 = e_2, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_1 = 0,$$
$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0.$$  

Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor $\bar{R}$ (up to symmetry and skew-symmetry)

$$\bar{R}(e_1, e_3, e_1, e_3) = \bar{R}(e_2, e_3, e_2, e_3) = 1 = \bar{R}(e_1, e_2, e_1, e_2).$$
Since \( \{e_1, e_2, e_3\} \) forms a basis, any vector field \( X, Y, U, V \in \chi(M) \) can be written as

\[
X = \sum_{i=1}^{3} a_i e_i, \quad Y = \sum_{i=1}^{3} b_i e_i, \quad U = \sum_{i=1}^{3} c_i e_i, \quad V = \sum_{i=1}^{3} d_i e_i,
\]

\[
\bar{R}(X,Y,U,V) = (a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1) + (a_1b_3 - a_3b_1)(c_1d_3 - c_3d_1) + (a_2b_3 - a_3b_2)(c_2d_3 - c_3d_2)
\]

\[
= T_1 \text{ (say)}
\]

\[
\bar{R}(e_1,Y,U,V) = b_3(c_1d_3 - c_3d_1) + b_2(c_1d_2 - c_2d_1) = \lambda_1 \text{ (say)}
\]

\[
\bar{R}(e_2,Y,U,V) = b_3(c_2d_3 - c_3d_2) - b_1(c_1d_2 - c_2d_1) = \lambda_2 \text{ (say)}
\]

\[
\bar{R}(e_3,Y,U,V) = b_1(c_3d_1 - c_1d_3) + b_2(c_2d_2 - c_3d_3) = \lambda_3 \text{ (say)}
\]

\[
\bar{R}(X,e_1,U,V) = a_3(c_1d_3 - c_3d_1) + a_2(c_1d_2 - c_2d_1) = \lambda_4 \text{ (say)}
\]

\[
\bar{R}(X,e_2,U,V) = a_3(c_2d_3 - c_3d_2) + a_1(c_2d_1 - c_1d_2) = \lambda_5 \text{ (say)}
\]

\[
\bar{R}(X,e_3,U,V) = a_1(c_3d_1 - c_1d_3) + a_2(c_3d_2 - c_2d_3) = \lambda_6 \text{ (say)}
\]

\[
\bar{R}(X,Y,e_1,V) = d_3(a_1b_3 - a_3b_1) + d_2(a_1b_2 - a_2b_1) = \lambda_7 \text{ (say)}
\]

\[
\bar{R}(X,Y,e_2,V) = d_3(a_2b_3 - a_3b_2) + d_1(a_2b_1 - a_1b_2) = \lambda_8 \text{ (say)}
\]

\[
\bar{R}(X,Y,e_3,V) = d_1(a_3b_1 - a_1b_3) + d_2(a_3b_2 - a_2b_3) = \lambda_9 \text{ (say)}
\]

\[
\bar{R}(X,Y,U,e_1) = c_3(a_1b_3 - a_3b_1) + c_2(a_1b_2 - a_2b_1) = \lambda_{10} \text{ (say)}
\]

\[
\bar{R}(X,Y,U,e_2) = a_3(c_2b_1 - a_1b_3) + c_1(a_2b_1 - a_1b_2) = \lambda_11 \text{ (say)}
\]

\[
\bar{R}(X,Y,U,e_3) = c_1(a_3b_1 - a_1b_3) + c_2(a_3b_2 - a_2b_3) = \lambda_12 \text{ (say)}
\]

\[
\bar{G}(X,Y,U,V) = (b_1c_1 + b_2c_2 - b_3c_3)(a_1d_1 + a_2d_2 - a_3d_3)
\]

and the components which can be obtained from these by the symmetry properties.

Now, we calculate the covariant derivatives of the non-vanishing components of the
curvature tensor as follows

\[(\nabla_{e_i} \bar{R})(X, Y, U, V) = -a_1\lambda_3 + a_2\lambda_2 - b_1\lambda_6 + b_3\lambda_5 - c_1\lambda_9 + c_3\lambda_8 - d_1\lambda_{12} + d_3\lambda_{11},\]

\[(\nabla_{e_2} \bar{R})(X, Y, U, V) = -a_2\lambda_3 + a_3\lambda_2 - b_2\lambda_6 + b_3\lambda_5 - c_2\lambda_9 + c_3\lambda_8 - d_2\lambda_{12} + d_3\lambda_{11},\]

\[(\nabla_{e_3} \bar{R})(X, Y, U, V) = 0.\]

Depending on the following choice of the 1-forms

\[A_1(e_1) = \frac{a_3\lambda_2 - a_1\lambda_3 + b_3\lambda_5 - b_1\lambda_6}{T_1},\]

\[A_2(e_1) = \frac{c_3\lambda_8 - c_1\lambda_9 + d_3\lambda_{11} - d_1\lambda_{12}}{T_2},\]

\[A_1(e_2) = \frac{a_3\lambda_2 - a_2\lambda_3 + b_3\lambda_5 - b_2\lambda_6}{T_1},\]

\[A_2(e_2) = \frac{c_3\lambda_8 - c_2\lambda_9 + d_3\lambda_{11} - d_2\lambda_{12}}{T_2},\]

\[B_1(e_3) = \frac{1}{a_3\lambda_3 + b_3\lambda_6},\]

\[B_2(e_3) = \frac{1}{a_3\theta_3 + b_3\theta_6},\]

\[D_1(e_3) = -\frac{1}{c_3\lambda_9 + d_3\lambda_{12}},\]

\[D_2(e_3) = -\frac{1}{c_3\theta_9 + d_3\theta_{12}}.\]

one can easily verify the relations

\[(\nabla_{e_i} \bar{R})(X, Y, U, V) = A_1(e_i)\bar{R}(X, Y, U, V) + B_1(X)\bar{R}(e_i, Y, U, V)
+ B_1(Y)\bar{R}(X, e_i, U, V) + D_1(U)\bar{R}(X, Y, e_i, V)
+ D_1(V)\bar{R}(X, Y, e_i, U) + A_2(e_i)\bar{G}(X, Y, U, V)
+ B_2(X)\bar{G}(e_i, Y, U, V) + B_2(Y)\bar{G}(X, e_i, U, V)
+ D_2(U)\bar{G}(X, Y, e_i, V) + D_2(V)\bar{G}(X, Y, U, e_i)\]

for 1, 2, 3. From the above, we can state that

**Theorem 5.1.** There exist a Kenmotsu manifold \((M^3, g)\) which is a generalized weakly symmetry Kenmotsu manifold.

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