On the Derivative of a Polynomial

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Abstract: Let \( P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}, \quad 1 \leq \mu < n, \) be a polynomial of degree at most \( n \) having no zeros in \(|z| < k, k \leq 1, \) and \( Q(z) = z^n P(1/z), \) it is proved by Dewan et al. \([5]\) that if \(|P'(z)|\) and \(|Q'(z)|\) becomes maximum at the same point on \(|z| = 1, \) then
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n-\mu+1}} \{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \}.
\]
In this paper, we generalize the above inequality for the polynomials of type \( P(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu}, \quad 1 \leq \mu \leq n. \)

Key Words: Polynomial, Inequality, Maximum modulus, Restricted zeros.

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1. Introduction and statement of results

Let \( P(z) \) be a polynomial of degree \( n, \) then according to the well known Bernstein’s inequality on the derivative of a polynomial, we have
\[
\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.
\] (1.1)
The result is best possible and equality holds for the polynomials having all its zeros at the origin.
For polynomials having no zeros in \(|z| < 1, \) Erdős conjectured and later Lax \([8]\) proved that if \( P(z) \neq 0 \) in \(|z| < 1, \) then (1.1) can be replaced by
\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|.
\] (1.2)
With equality for those polynomials, which have all their zeros on \(|z| = 1. \)
In the literature, there already exists various refinements and generalizations of
(1.2), for example (see Aziz [1], Bidkham et al. [2,3,4], Khojastehnezhad and Bidkham [7], Zireh [14] etc). As an extension of (1.2) Malik [12] proved that if \( P(z) \neq 0 \) in \( |z| < k, k \geq 1 \), then

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k} \max_{|z|=1} |P(z)|. \tag{1.3}
\]

Further Govil [9] proved that for the polynomial \( P(z) = \sum_{j=0}^{n} a_j z^j \) which has no zeros in \( |z| < k, k \leq 1 \), if \( |P'(z)| \) and \( |Q'(z)| \) becomes maximum at the same point on \( |z| = 1 \), then

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k} \max_{|z|=1} |P(z)|. \tag{1.4}
\]

Whereas the polynomial \( P(z) = \sum_{j=0}^{n} a_j z^j \) having all its zeros on \( |z| = k, k \leq 1 \), Govil [10] proved

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |P(z)|. \tag{1.5}
\]

Recently Dewan and Hans [5] obtained a generalization of (1.4) and proved for \( P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}, 1 \leq \mu < n \) that having no zeros in \( |z| < k, k \leq 1 \), if \( |P'(z)| \) and \( |Q'(z)| \) becomes maximum at the same point on \( |z| = 1 \), then

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n-\mu+1}} \{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \}. \tag{1.6}
\]

For \( P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}, 1 \leq \mu < n \) that having all its zeros on \( |z| = k, k \leq 1 \), Dewan [5] also proved

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |P(z)|. \tag{1.7}
\]

In this paper, first we obtain the following result

**Theorem 1.1.** Let \( P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu \), \( 1 \leq \mu \leq n \) is a polynomial of degree \( n \), having no zeros in \( |z| < k, k \leq 1 \) and \( Q(z) = z^n P(1/z) \). If \( |P'(z)| \) and \( |Q'(z)| \) becomes maximum at the same point on \( |z| = 1 \), then

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} \max_{|z|=1} |P(z)|. \tag{1.8}
\]

**Remark 1.2.** If we take \( \mu = 1 \) in Theorem 1.1, then inequality (1.8) reduces to inequality (1.4) due to Govil.

Next we prove the following interesting result which is a refinement of inequality (1.8).

**Theorem 1.3.** Let \( P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu \), \( 1 \leq \mu \leq n \) is a polynomial of degree \( n \), having no zeros in \( |z| < k, k \leq 1 \) and \( Q(z) = z^n P(1/z) \). If \( |P'(z)| \) and \( |Q'(z)| \) becomes maximum at the same point on \( |z| = 1 \), then

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} \{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \}. \tag{1.9}
\]
Remark 1.4. If we take \( \mu = 1 \) in Theorem 1.3, then inequality (1.9) reduces to the following result which proved by Aziz and Ahmad [1].

Corollary 1.5. Let \( P(z) \) is a polynomial of degree \( n \), having no zeros in \( |z| < k \), \( k \leq 1 \) and \( Q(z) = z^n P(1/z) \). If \( |P'(z)| \) and \( |Q'(z)| \) becomes maximum at the same point on \( |z| = 1 \), then

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + kn} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right\}. \tag{1.10}
\]

Finally we prove the following result.

Theorem 1.6. Let \( P(z) = a_0 + \sum_{v=0}^{n} a_v z^v \), \( 1 \leq \mu \leq n \) is a polynomial of degree \( n \), having all its zeros on \( |z| = k \), \( k \leq 1 \), then

\[
\max_{|z|=1} |P'(z)| \leq \frac{n}{kn+\mu-1 + kn+\mu-2} \max_{|z|=1} |P(z)|. \tag{1.11}
\]

Remark 1.7. If we take \( \mu = 1 \) in Theorem 1.6, then inequality (1.11) reduces to inequality (1.5) due to Govil.

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 2.1. [13] Let \( P(z) \) be a polynomial of degree \( n \), then for \( R \geq 1 \)

\[
\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \tag{2.1}
\]

Lemma 2.2. Let \( P(z) = c_n z^n + \sum_{v=\mu}^{n} c_v z^{n-v} \), \( 1 \leq \mu \leq n \) be a polynomial of degree \( n \), having all its zeros in \( |z| \leq k \), \( k \geq 1 \), then for \( |z| = 1 \)

\[
k^{n-\mu-1} |Q'(z)| \leq |P'(k^2 z)|, \tag{2.2}
\]

where \( Q(z) = z^n P(1/z) \).

Proof: Let \( F(z) = P(kz) \), then \( F(z) \) has all its zeros in \( |z| \leq 1 \). If \( G(z) = z^n F(1/z) = z^n P(k/z) = k^n Q(z/k) \), then all the zeros of \( G(z) \) lie in \( |z| \geq 1 \). Since \( |F(z)| = |G(z)| \) on \( |z| = 1 \), we can say that an application of maximum modulus principle to the function \( \frac{G(z)}{F(z)} \) will yield \( |G(z)| \leq |F(z)|, |z| \geq 1 \). Therefore the polynomial \( G(z) = \lambda F(z) \), will not vanish in \( |z| > 1 \) for every \( \lambda \) with \( |\lambda| > 1 \). Gauss-Lucas theorem will then imply that polynomial \( G'(z) - \lambda F'(z) \) will not vanish in \( |z| > 1 \) for every \( \lambda \) with \( |\lambda| > 1 \) and therefore \( |G'(z)| \leq |F'(z)|, |z| \geq 1 \). Substituting for \( F'(z) \) and \( G'(z) \), we get

\[
k^{n-\mu-1} |Q'(z/k)| \leq k|P'(kz)|, \tag{2.3}
\]
where \( |z| \geq 1. \)
Since \( Q(z) = \tau_n + \sum_{\nu=\mu}^{n} \tau_{n-\nu} z^\nu \), then
\[
k^{n-1} \left| \sum_{\nu=\mu}^{n} \nu \tau_{n-\nu} (\frac{z}{k})^{\nu-1} \right| \leq k|P'(kz)|.
\]
i.e,
\[
k^{n-\mu} \left| \sum_{\nu=\mu}^{n} \nu \tau_{n-\nu} (\frac{z}{k})^{\nu-\mu} \right| \leq k|P'(kz)|,
\]
where \( |z| \geq 1. \)
If we take \( kz \) instead of \( z \) in inequality (2.4), then we have
\[
k^{n-\mu} \left| \sum_{\nu=\mu}^{n} \nu \tau_{n-\nu} z^{\nu-\mu} \right| \leq k|P'(k^2 z)|,
\]
where \( |z| \geq 1/k. \)
Since \( 1/k \leq 1, \) we have in particular,
\[
k^{n-\mu} \left| \sum_{\nu=\mu}^{n} \nu \tau_{n-\nu} z^{\nu-\mu} \right| \leq k|P'(k^2 z)|,
\]
where \( |z| \geq 1. \)
This implies
\[
k^{n-\nu} \left| \sum_{\nu=\mu}^{n} \nu \tau_{n-\nu} z^{\nu-1} \right| \leq k|P'(k^2 z)|,
\]
where \( |z| = 1. \)
This completes the proof of Lemma 2.2. \( \Box \)

Lemma 2.3. Let \( P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}, \ 1 \leq \mu \leq n \) be a polynomial of degree \( n \), having all its zeros in \( |z| \leq k, \ k \geq 1, \) then
\[
\max_{|z|=1} |Q'(z)| \leq k^{n+\mu-1} \max_{|z|=1} |P'(z)|,
\]
where \( Q(z) = z^n P(1/z). \)

Proof: On applying Lemma 2.2 we have
\[
k^{n-\nu} |Q'(z)| \leq |P'(k^2 z)|.
\]
Now using Lemma 2.1 for the polynomial \( P'(k^2 z) \), of degree \( n-1 \). We have
\[
\max_{|z|=k^2} |P'(z)| \leq k^{2n-2} \max_{|z|=1} |P'(z)|.
\]
Combining (2.9) and (2.10), we have desired result. \( \Box \)
Lemma 2.4. Let \( P(z) = a_0 + \sum_{\nu=1}^{n} a_{\nu} z^\nu \), \( 1 \leq \mu \leq n \) be a polynomial of degree \( n \), has no zeros in \( |z| < k \), \( k \leq 1 \), then

\[
k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq \max_{|z|=1} |Q'(z)|,
\]

(2.11)

where \( Q(z) = z^n P\left(\frac{1}{z}\right) \).

Proof: Since \( P(z) \) has no zeros in \( |z| < k \), then \( Q(z) = z^n P\left(\frac{1}{z}\right) \) has all its zeros in \( |z| \leq 1/k, 1/k \geq 1 \). On applying Lemma 2.3 to the polynomial \( Q(z) \), we have

\[
k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq \max_{|z|=1} |Q'(z)|.
\]

\[ \square \]

The following lemma is due to Malik [6].

Lemma 2.5. Let \( P(z) \) be a polynomial of degree \( n \), has no zero in \( |z| < k \), \( k \geq 1 \), then for \( |z| = 1 \)

\[
k |P'(z)| \leq |Q'(z)|
\]

(2.12)

where \( Q(z) = z^n P\left(\frac{1}{z}\right) \).

Lemma 2.6. Let \( P(z) \) be a polynomial of degree \( n \), having all its zeros on \( |z| = k \), \( k \leq 1 \), then for \( |z| = 1 \)

\[
|Q'(z)| \leq k |P'(z)|
\]

(2.13)

where \( Q(z) = z^n P\left(\frac{1}{z}\right) \).

Proof: Since \( P(z) \) has all its zeros on \( |z| = k \), then \( Q(z) = z^n P\left(\frac{1}{z}\right) \) has all its zeros in \( |z| = 1/k, 1/k \geq 1 \). On applying Lemma 2.5 to the polynomial \( Q(z) \), we have

\[
1/k |Q'(z)| \leq |P'(z)|.
\]

\[ \square \]

The following lemma is a special case of a result due to Govil and Rahman [11].

Lemma 2.7. Let \( P(z) \) be a polynomial of degree \( n \), then for \( |z| = 1 \)

\[
|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|
\]

(2.14)

where \( Q(z) = z^n P\left(\frac{1}{z}\right) \).
3. Proofs of the theorems

Proof of Theorem 1.1. Since $|P'(z)|$ and $|Q'(z)|$ attained maximum at the same point on $|z| = 1$. This implies there exist a point $z_0$ such that $|P'(z_0)| = \max_{|z|=1} |P'(z)| = \max_{|z|=1} |Q'(z)| = |Q'(z_0)|$. On the other hand by Lemma 2.7, we have

$$|P'(z_0)| + |Q'(z_0)| \leq n \max_{|z|=1} |P(z)|.$$ 

On applying Lemma 2.4, we have

$$|P'(z_0)| + k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$ 

This implies

$$\max_{|z|=1} |P'(z)| + k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$ 

This completes the proof of Theorem 1.1. \qed

Proof of Theorem 1.3. Let $m = \min_{|z|=k} |P(z)|$. For $\alpha$ with $|\alpha| < 1$, we have $|\alpha m| < m \leq |P(z)|$, where $|z| = k$. Therefore by implying Rouche’s theorem, the polynomial $G(z) = P(z) - \alpha m$ has no zeros in $|z| < k$. On applying Theorem 1.1 to the polynomial $G(z)$, we have

$$\max_{|z|=1} |G'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} \max_{|z|=1} |G(z)|,$$

i.e.,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} \max_{|z|=1} |P(z) - \alpha m|.$$ 

If we choose a point $z_0$ on $|z| = 1$ such that $\max_{|z|=1} |P(z)| = |P(z_0)|$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} |P(z_0) - \alpha m|.$$ 

Now by suitable choice of argument of $\alpha$, we get

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^{n+\mu-1}} (|P(z_0)| - |\alpha|m).$$

By making $|\alpha| \to 1$, the result follows. \qed

Proof of Theorem 1.6. If $z_0$ is a point on $|z| = 1$ such that $|Q'(z_0)| = \max_{|z|=1} |Q'(z)|$. Then by Lemma 2.7, we have

$$|P'(z_0)| + |Q'(z_0)| \leq n \max_{|z|=1} |P(z)|.$$ 

On applying Lemma 2.6, we have

$$\frac{1}{k} |Q'(z_0)| + |Q'(z_0)| \leq n \max_{|z|=1} |P(z)|.$$
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\[
\left( \frac{1}{k} + 1 \right) \max_{|z|=1} |Q'(z)| \leq n \max_{|z|=1} |P(z)|.
\]

Now applying Lemma 2.4, we have

\[
\left( \frac{1}{k} + 1 \right) k^{n+\mu-1} \max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.
\]

This completes the proof of Theorem 1.6. \(\square\)

Acknowledgments

The authors wish to sincerely thank the referees, for the careful reading of the paper and for the helpful suggestions and comments.

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