Darboux Curves in Lorentzian Three Dimensional Heisenberg Group

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Abstract: In this paper, a new characterization for darboux curves in $\text{Heis}_3$ is completely given. Then, a new classification for translation surface, which is generated by darboux curve in $\text{Heis}_3$ is obtained.

Key Words: Heisenberg group, Lorentz metric.

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1. Introduction

Darboux curves firstly were studied by Saban [17] in the Euclidean space, thereafter generalised by Ergin [2]. For a curve $\alpha$ on a surface in the $E^3$ the function

$$D = \langle \alpha''', u \rangle = \kappa_u' - \kappa_g \tau_g$$

is called Darboux function of $\alpha$. Here $u$ is normal field of surface, $\kappa_u, \kappa_g$ and $\tau_g$ are normal curvature, geodesic curvature and geodesic torsion. If Darboux function is equal to zero, then these curves called Darboux curves. In Minkowski 3-space timelike Darboux curves on a timelike surface were studied by Ergin [3].

Translation surfaces in $E^3$, firstly studied by H. F. Scherk. He showed that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{a} \log \frac{\cos (ax)}{\cos (ay)} = \frac{1}{a} \log |\cos (ax)| - \frac{1}{a} \log |\cos (ay)|,$$

where $a$ is a non-zero constant, [6]. Then, when the second fundamental form was considered as a metric on a non-developable surface, translation surfaces in the Euclidean space were obtained by, [15]. The translation surfaces which are generated by two space curves in $E^3$ have been investigated by Çetin. Also they showed that Scherk surface is not only minimal translation surface, [1] D. W. Yoon has studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map $G$ satisfies the condition $\Delta G = \Delta A$, where $\Delta$ denotes the Laplacian of the
surface, [19]. Translation surfaces in terms of a pair of two planar curves lying in orthogonal planes defined by [6] in the $\text{Nil}^3$ with left invariant Riemannian metric. They classified minimal translation surfaces in $\text{Nil}^3$. Translation surfaces in $\text{Sol}^3$ constructed by [13] and they investigated properties of minimal one. Also some curves and surfaces studied in [7-12].

The purpose of this paper is to study and classify modified translation surfaces in $\text{Heis}_3$ and investigate conditions of being minimal surface. Also, obtain characterizations of points on this surface.

2. The Heisenberg Group

The Heisenberg group $\text{Heis}_3$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) \ast (x_1, y_1, z_1) = \left( x + x_1, y + y_1, z + z_1 + \frac{1}{2} (xy_1 - x_1y) \right).$$ (2.1)

The left-invariant Lorentzian metric on $\text{Heis}_3$ is

$$g = ds^2 = -dx^2 + dy^2 + (xdy + dz)^2. \quad (2.2)$$

The orthonormal basis for the corresponding Lie algebra:

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x}. \quad (2.3)$$

Then, we have

$$[e_2, e_3] = 2e_1, \quad [e_1, e_2] = [e_1, e_3] = 0 \quad (2.4)$$

with

$$g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1. \quad (2.5)$$

**Proposition 2.1.** For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g$ is

$$\nabla_{e_i} e_j = \begin{bmatrix} 0 & e_3 & e_2 \\ e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{bmatrix},$$ (2.6)

where the $(i, j)$-element in the table above equals for $\nabla_{e_i} e_j$ for our basis

$$\{e_k, k = 1, 2, 3\} = \{e_1, e_2, e_3\}.$$

Let $\gamma : I \rightarrow \text{Heis}_3$ be a unit speed spacelike curve with timelike binormal and $\{T, N, B\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\nabla_T T = \kappa N, \quad \nabla_T N = -\kappa T + \tau B \quad (2.7)$$

$$\nabla_T B = \tau N,$$
where $\kappa$, $\tau$ are curvature function and torsion function. With respect to the orthonormal basis $\{e_1, e_2, e_3\}$, we can write
\[
T = t_1 e_1 + t_2 e_2 + t_3 e_3, \\
N = n_1 e_1 + n_2 e_2 + n_3 e_3, \\
B = b_1 e_1 + b_2 e_2 + b_3 e_3.
\] (2.9)

3. Darboux Curves in 3-D Lorentzian Heisenberg Group

Let unit tangent vector field of the curve be
\[
T = t_1 e_1 + t_2 e_2 + t_3 e_3
\]
and the unit normal vector field of the surface be
\[
u = u_1 e_1 + u_2 e_2 + u_3 e_3.
\]

Theorem 3.1. Let $\alpha : I \rightarrow M$ be a unit speed curve in $(\text{Heis}_3, g)$. If $\alpha$ is a darboux curve on surface $M$, then
\[
\delta' u_1 t_1 + \lambda' u_2 t_2 + \gamma' u_3 t_1 + \frac{1}{2} (u_1 (t_2 \gamma + t_3 \lambda) - u_2 (t_1 \gamma + t_3 \delta) + u_3 (t_1 \lambda - t_2 \delta)) = 0.
\]

Proof. If we notice
\[
\delta = t_1', \\
\lambda = t_2' + 2t_1 t_3, \\
\gamma = t_3' + 2t_1 t_2,
\] (3.1)
then we have
\[
\nabla_T (\nabla_T T) = (t_1'' + t_2' (t_3' + 2t_1 t_2) - t_3 (t_2' + 2t_1 t_3)) e_1 \\
+ (t_2'' + 2(t_1 t_3)' + t_1 (t_3' + 2t_1 t_2) - t_3 t_1') e_2 \\
+ (t_3'' + 2(t_1 t_2)' - t_1 (t_2' + 2t_1 t_3) + t_2 t_1') e_3.
\] (3.2)

So, the darboux function is
\[
D = g (\nabla_T (\nabla_T T), u)
= (t_1'' + t_2' (t_3' + 2t_1 t_2) - t_3 (t_2' + 2t_1 t_3) u_1 \\
+ (t_2'' + 2(t_1 t_3)' + t_1 (t_3' + 2t_1 t_2) - t_3 t_1') u_2 \\
- (t_3'' + 2(t_1 t_2)' - t_1 (t_2' + 2t_1 t_3) + t_2 t_1') u_3.
\] (3.3)
If $\alpha$ is a darboux curve on surface $M$, from the equation (3.3) we have

$$
(t''_1 + 2(t'_2 + 2t_1t_2) - t_3(t'_3 + 2t_1t_3)u_1 \\
+ (t''_2 + 2(t'_1t_3)' + t_1 (t'_2 + 2t_1t_2) - t_3t'_1)u_2 \\
- (t''_3 + 2(t'_1t_2)' - t_1 (t'_2 + 2t_1t_3) + t_2t'_1)u_3 = 0.
$$

(3.4)

**Corollary 3.2.** Let $\alpha$ be an unit speed spacelike curve and $M$ be a spacelike ruled surface in $(Heis_3,g)$ which is parametrized as

$$
M(x,y) = \alpha(x) + y\mathbf{T}(x).
$$

(3.5)

If $\alpha$ is a darboux curve, then

$$
(t''_1 + t_2 (t'_3 + 2t_1t_2) - t_3(t'_3 + 2t_1t_3)(t_2 (t'_2 + 2t_1t_2) - t_3(t'_2 + 2t_1t_3)) \\
+ (t''_2 + 2(t'_1t_3)' + t_1 (t'_2 + 2t_1t_2) - t_3t'_1)(t_3t'_1 - t_1 (t'_3 + 2t_1t_2)) \\
- (t''_3 + 2(t'_1t_2)' - t_1 (t'_2 + 2t_1t_3) + t_2t'_1)(t_2t'_1 - t_1 (t'_2 + 2t_1t_3)) = 0.
$$

(3.6)

**Proof.** From equation (3.5), we have

$$
M_x(x,y) = \mathbf{T}(x) + y\nabla_T \mathbf{T}
$$

(3.7)

$$
M_y(x,y) = \mathbf{T}(x).
$$

(3.8)

From equations (3.6)-(3.7), then the unit normal vector field of the surface $M$ is

$$
\mathbf{u} = \frac{1}{\omega} \{(t_2 (t'_3 + 2t_1t_2) - t_3(t'_3 + 2t_1t_3))\mathbf{e}_1 \\
+ (t_3t'_1 - t_1 (t'_3 + 2t_1t_2))\mathbf{e}_2 \\
+ (t_2t'_1 - t_1 (t'_2 + 2t_1t_3))\mathbf{e}_3\}.
$$

(3.9)

where

$$
\omega = (t_2 (t'_3 + 2t_1t_2) - t_3(t'_3 + 2t_1t_3))^2 + (t_3t'_1 - t_1 (t'_3 + 2t_1t_2))^2 \\
- (t_2t'_1 - t_1 (t'_2 + 2t_1t_3))^2.
$$

If $\alpha$ is a darboux curve, from equations (3.2), (3.3) and (3.9), we have (3.6).

**Example 3.3.** In $(Heis_3,g)$

$$
\alpha(x) = (c_1, \frac{x^2}{2} + c_2, x^2 - \frac{c_1x^2}{2} + c_3)
$$

(3.10)

is a unit speed curve where $c_i$ ($i = 1, 2, 3$) are constant. The unit spacelike tangent vector field of the $\alpha$ is

$$
\mathbf{T}(x) = (0, x, 2x)
$$

(3.11)
From equations (3.10), (3.11), the ruled surface which is parametrized (3.5) is

\[ M(x, y) = (c_1, \frac{x^2}{2} + xy + c_2, x^2 - \frac{c_1 x^2}{2} + 2xy + c_3). \]

The unit normal vector field of the \( M(x, y) \) is

\[ \mathbf{u} = \frac{1}{\omega}(0, -2x(c_1 x + 2x)^2, 2x^2(c_1 x + 2x)(1 - c_1 x - 2x)). \]

References


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