Initial Coefficient Bounds for a Subclass of m-fold Symmetric Bi-univalent Functions

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ABSTRACT: In this paper, we introduce and investigate a subclass $B_{h,p}^{\Sigma_m}(\tau, \lambda)$ of analytic and bi-univalent functions which both $f(z)$ and $f^{-1}(z)$ are m-fold symmetric in the open unit disk $U$. Furthermore, we find upper bounds for the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in this subclass. The results presented in this paper would generalize and improve some recent works.

Key Words: Analytic functions, Bi-univalent functions, Coefficient estimates, m-fold symmetric functions, m-fold symmetric bi-univalent functions.

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1. Introduction

Let $\mathcal{A}$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ which are univalent in $U$.

The Koebe one-quarter Theorem [4] ensures that the image of $U$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \ (z \in U),$$

and

$$f(f^{-1}(w)) = w \ \left( |w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

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where
\[ f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \] (1.2)

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \). Let \( \Sigma \) denote the class of bi-univalent functions in \( U \) given by (1.1).

Lewin [8] investigated the class \( \Sigma \) of bi-univalent functions and showed that \( |a_2| < 1.51 \) for the functions belonging to \( \Sigma \). Subsequently, Brannan and Clunie [1] conjectured that \( |a_2| \leq \sqrt{2} \). Kedzierawski [6] proved this conjecture for a special case when the function \( f \) and \( f^{-1} \) are starlike functions. Recently there interest to study the bi-univalent functions class \( \Sigma \) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) ([2,3,10,11,12,16,17]). The coefficient estimate problem i.e. bound of \( |a_n| \) (\( n \in \mathbb{N} - \{2,3\} \)) for each \( f \in \Sigma \) is still an open problem.

For each function \( f \in \mathcal{S} \), the function \( h(z) = \sqrt{f(z)} \) \( (z \in U, m \in \mathbb{N}) \), is univalent and maps the unit disk \( U \) into a region with \( m \)-fold symmetry. A function is said to be \( m \)-fold symmetric (see [7,9]) if it has the following normalized form:
\[ f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \] (1.3)

We denote by \( \mathcal{S}_m \) the class of \( m \)-fold symmetric univalent functions in \( U \), which are normalized by the series expansion (1.3). In fact, the functions in the class \( \mathcal{S} \) are one-fold symmetric.

Analogous to the concept of \( m \)-fold symmetric univalent functions, we here introduced the concept of \( m \)-fold symmetric bi-univalent functions. Each function \( f \in \Sigma \) generates an \( m \)-fold symmetric bi-univalent function for each integer \( m \in \mathbb{N} \). The normalized form of \( f \) is given as in (1.3) and the series expansion for \( f^{-1} \) which was recently proven by Srivastava et al. [13], is given as follows:
\[ g(w) = w - a_{m+1} w^{m+1} + [(m + 1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} \]
\[ - \left[ \frac{1}{2} (m + 1)(3m + 2)a_{m+1}^3 - (3m + 2)a_{m+1}a_{2m+1} + a_{3m+1} \right] w^{3m+1} + \ldots, \] (1.4)

where \( g = f^{-1} \). We denote by \( \Sigma_m \) the class of \( m \)-fold symmetric bi-univalent functions in \( U \). For \( m = 1 \), the formula (1.4) coincides with the formula (1.2) of the class \( \Sigma \). Some examples of \( m \)-fold symmetric bi-univalent functions are given as follows:
\[ \left( \frac{z^m}{1 - z^m} \right)^{\frac{1}{m}} \text{ and } [-\log(1 - z^m)]^{\frac{1}{m}} \].
with the corresponding inverse functions given by
\[
\left( \frac{w^m}{1-w^m} \right)^\frac{1}{m} \quad \text{and} \quad \left( \frac{e^{w^m}-1}{e^{w^m}} \right)^\frac{1}{m}
\]
respectively.

Recently, Srivastava et al. [14] investigated the following two subclasses \( \mathcal{B}_{\Sigma_m}(\tau, \lambda, \alpha) \) and \( \mathcal{B}_{\Sigma_m}^*(\tau, \lambda, \beta) \) of \( \Sigma_m \) consisting of m-fold symmetric bi-univalent functions in the open unit disk \( U \) and obtain coefficient bounds for \( |a_{m+1}| \) and \( |a_{2m+1}| \) for functions in each of these new subclasses.

**Definition 1.1.** (see [14]) Let \( 0 < \alpha \leq 1, \tau \in \mathbb{C} \setminus \{0\} \) and \( \lambda \geq 1 \). A function \( f(z) \) given by (1.3) is said to be in the class \( \mathcal{B}_{\Sigma_m}(\tau, \lambda, \alpha) \) if the following conditions are satisfied:

\[
f \in \Sigma_m \quad \text{and} \quad \left| \arg \left( 1 + \frac{1}{\tau} \left[ (1-\lambda) \frac{f(z)}{z} + \lambda f'(z) - 1 \right] \right) \right| < \frac{\alpha \pi}{2} \quad (z \in U),
\]

and

\[
\left| \arg \left( 1 + \frac{1}{\tau} \left[ (1-\lambda) \frac{g(w)}{w} + \lambda g'(w) - 1 \right] \right) \right| < \frac{\alpha \pi}{2} \quad (w \in U),
\]

where the function \( g \) is given by (1.4).

**Theorem 1.2.** (see [14]) Let the function \( f(z) \) given by (1.3) be in the class \( \mathcal{B}_{\Sigma_m}(\tau, \lambda, \alpha) \). Then

\[
|a_{m+1}| \leq 2\alpha \left| \frac{\tau}{m+1}(2\lambda m+1) + (1-\alpha)(\lambda m+1)^2 \right|
\]

and

\[
|a_{2m+1}| \leq 2\alpha^2 \left| \frac{\tau}{m+1}(2\lambda m+1) \right| + \frac{2\alpha |\tau|}{2\lambda m+1}.
\]

**Definition 1.3.** (see [14]) Let \( 0 \leq \beta < 1, \tau \in \mathbb{C} \setminus \{0\} \) and \( \lambda \geq 1 \). A function \( f(z) \) given by (1.3) is said to be in the class \( \mathcal{B}_{\Sigma_m}^*(\tau, \lambda, \beta) \) if the following conditions are satisfied:

\[
f \in \Sigma_m \quad \text{and} \quad \Re \left( 1 + \frac{1}{\tau} \left[ (1-\lambda) \frac{f(z)}{z} + \lambda f'(z) - 1 \right] \right) > \beta \quad (z \in U),
\]

and

\[
\Re \left( 1 + \frac{1}{\tau} \left[ (1-\lambda) \frac{g(w)}{w} + \lambda g'(w) - 1 \right] \right) > \beta \quad (w \in U),
\]

where the function \( g \) is given by (1.4).

**Theorem 1.4.** (see [14]) Let the function \( f(z) \) given by (1.3) be in the class \( \mathcal{B}_{\Sigma_m}^*(\tau, \lambda, \beta) \). Then

\[
|a_{m+1}| \leq \sqrt{\frac{4|\tau|(1-\beta)}{(m+1)(2\lambda m+1)}}.
\]
and

\[ |a_{2m+1}| \leq \frac{2|\tau|^2(1-\beta)^2(m+1)}{(\lambda m + 1)^2} + \frac{2|\tau|(1-\beta)}{2\lambda m + 1}. \]

The objective of the present paper is to introduce a formula for computing the coefficients \( |a_{m+1}| \) and \( |a_{2m+1}| \) for functions in each of these new subclasses which improve the coefficient bounds obtained in Theorem 1.2 and Theorem 1.4. Our results generalize and improve some recent works as Srivastava [11,13,14], Eker [15] and Frasin and Aouf [5].

2. The subclass \( B_{\Sigma_m}^{h,p}(\tau, \lambda) \)

In this section, we introduce and investigate the general subclass \( B_{\Sigma_m}^{h,p}(\tau, \lambda) \).

Definition 2.1. Let the functions \( h, p : U \to \mathbb{C} \) be analytic functions and

\[
h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots,
\]

\[
p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots,
\]

such that

\[
\min\{\Re(h(z)), \Re(p(z))\} > 0 \quad (z \in U).
\]

Let \( \tau \in \mathbb{C} \setminus \{0\} \) and \( \lambda \geq 1 \). A function \( f \) given by (1.3) is said to be in the class \( B_{\Sigma_m}^{h,p}(\tau, \lambda) \) if the following conditions are satisfied:

\[
f \in \Sigma_m \quad \text{and} \quad 1 + \frac{1}{\tau} \left[ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) - 1 \right] \in h(U) \quad (z \in U), \quad (2.1)
\]

and

\[
1 + \frac{1}{\tau} \left[ (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) - 1 \right] \in p(U) \quad (w \in U), \quad (2.2)
\]

where the function \( g \) is defined by (1.4).

Remark 2.2. There are many choices of \( h \) and \( p \) which would provide interesting subclasses of class \( B_{\Sigma_m}^{h,p}(\tau, \lambda) \). For example,

1. For \( h(z) = p(z) = \left( \frac{1 + z^m}{1 + z} \right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots \), where \( 0 < \alpha \leq 1 \), it is easy to verify that the functions \( h(z) \) and \( p(z) \) satisfy the hypotheses of Definition 2.1. Now if \( f \in B_{\Sigma_m}^{h,p}(\tau, \lambda) \), then

\[
\left| \arg \left( 1 + \frac{1}{\tau} \left[ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) - 1 \right] \right) \right| < \frac{\alpha \pi}{2} \quad (z \in U),
\]

and

\[
\left| \arg \left( 1 + \frac{1}{\tau} \left[ (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) - 1 \right] \right) \right| < \frac{\alpha \pi}{2} \quad (w \in U).
\]

Therefore in this case, the class \( B_{\Sigma_m}^{h,p}(\tau, \lambda) \) reduce to class \( B_{\Sigma_m}(\tau, \lambda, \alpha) \) in Definition 1.1.
2. For $h(z) = p(z) = \frac{1 + (1 - 2\beta)z^m}{1 - z^m} = 1 + 2(1 - \beta)z^m + 2(1 - \beta)z^{2m} + \cdots$, where $0 \leq \beta < 1$, the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. Now if $f \in B_{\Sigma^m}(\tau, \lambda)$, then

$$\text{Re} \left( 1 + \frac{1}{\tau} \left[ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) - 1 \right] \right) > \beta \quad (z \in U),$$

and

$$\text{Re} \left( 1 + \frac{1}{\tau} \left[ (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) - 1 \right] \right) > \beta \quad (w \in U).$$

Therefore in this case, the class $B_{\Sigma^m}(\tau, \lambda)$ reduce to class $B_{\Sigma^m}(\tau, \lambda, \beta)$ in Definition 1.3.

3. Coefficient Estimates

Now, we obtain the estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for subclass $B_{\Sigma^m}(\tau, \lambda)$.

**Theorem 3.1.** Let the function $f(z)$ given by (1.3) be in the class $B_{\Sigma^m}(\tau, \lambda)$. Then

$$|a_{m+1}| \leq \min \left\{ \sqrt{\frac{\tau^2(|h^{(m)}(0)|^2 + |p^{(m)}(0)|^2)}{2(m!)^2(\lambda m + 1)^2}}, \frac{\tau(|h^{(2m)}(0)| + |p^{(2m)}(0)|)}{(2m!)(2\lambda m + 1)(m + 1)} \right\},$$

(3.1)

and

$$|a_{2m+1}| \leq \min \left\{ \frac{|\tau||h^{(2m)}(0)| + |p^{(2m)}(0)||}{2(2m)!((2\lambda m + 1)^2)}, \frac{|\tau|^2(m + 1)(|h^{(m)}(0)|^2 + |p^{(m)}(0)|^2)}{4(m!)^2(\lambda m + 1)^2}, \frac{|\tau||h^{(2m)}(0)|}{(2m!)(2\lambda m + 1)} \right\}.$$  

(3.2)

**Proof:** First of all, we write the argument inequalities in (2.1) and (2.2) in their equivalent forms as follows:

$$1 + \frac{1}{\tau} \left[ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) - 1 \right] = h(z) \quad (\lambda \geq 1, z \in U),$$

(3.3)

and

$$1 + \frac{1}{\tau} \left[ (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) - 1 \right] = p(w) \quad (\lambda \geq 1, w \in U),$$

(3.4)
respectively, where functions $h$ and $p$ satisfy the conditions of Definition 2.1. Also, the functions $h$ and $p$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots,$$  \hspace{1cm} (3.5)\\

and

$$p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots.$$  \hspace{1cm} (3.6)

Now, upon substituting from (3.5) and (3.6) into (3.3) and (3.4), respectively, and equating the coefficients, we get

$$(\lambda m + 1 + \frac{1}{\tau}) a_{m+1} = h_m,$$  \hspace{1cm} (3.7)\\

$$(2\lambda m + 1 + \frac{1}{\tau}) a_{2m+1} = h_{2m},$$  \hspace{1cm} (3.8)\\

and

$$-(\lambda m + 1 + \frac{1}{\tau}) a_{m+1} = p_m.$$  \hspace{1cm} (3.9)

From (3.7) and (3.9), we get

$$h_m = -p_m,$$  \hspace{1cm} (3.11)

and

$$2 \left(\frac{\lambda m + 1}{\tau}\right)^2 a_{m+1}^2 = h_m^2 + p_m^2.$$  \hspace{1cm} (3.12)

Adding (3.8) and (3.10), we get

$$\left(\frac{2\lambda m + 1}{\tau}\right)(m+1) a_{m+1}^2 = p_{2m} + h_{2m}.$$  \hspace{1cm} (3.13)

Therefore, from (3.12) and (3.13), we have

$$a_{m+1}^2 = \frac{\tau^2 (h_m^2 + p_m^2)}{2(\lambda m + 1)^2},$$  \hspace{1cm} (3.14)\\

and

$$a_{m+1}^2 = \frac{\tau (p_{2m} + h_{2m})}{2(\lambda m + 1)(m+1)},$$  \hspace{1cm} (3.15)

respectively. Therefore, we find from the equations (3.14) and (3.15), that

$$|a_{m+1}|^2 \leq \frac{\tau^2 (|h_m(0)|^2 + |p_m(0)|^2)}{2(m!)^2(\lambda m + 1)^2},$$
and
\[ |a_{m+1}|^2 \leq \frac{|\tau|(|h^{(2m)}(0)| + |p^{(2m)}(0)|)}{(2m)!} \frac{(m+1)!}{(2\lambda m + 1)!} \]
respectively. So we get the desired estimate on the coefficient \(|a_{m+1}|\) as asserted in (3.1).

Next, in order to find the bound on the coefficient \(|a_{2m+1}|\), by subtracting (3.10) from (3.8), we get
\[ 2 \left( \frac{2\lambda m + 1}{\tau} \right) a_{2m+1} = \left( \frac{2\lambda m + 1}{\tau} \right) (m+1) a_{2m+1} = h_{2m} - p_{2m}. \quad (3.16) \]

Upon substituting the value of \(a_{2m+1}^2\) from (3.14) into (3.16), it follows that
\[ a_{2m+1} = \tau^2 (m+1)(h_m^2 + p_m^2) \frac{(m+1)!}{(2\lambda m + 1)!}. \]

Therefore, we get
\[ |a_{2m+1}| \leq \frac{|\tau|^2 (m+1)(h^{(m)}(0))^2 + |p^{(m)}(0)|^2)}{4(m)!^2(\lambda m + 1)^2} \]
\[ + \frac{|\tau|(|h^{(2m)}(0)| + |p^{(2m)}(0)|)}{2(2m)!} \cdot \frac{(2\lambda m + 1)!}{(2m)!}. \quad (3.17) \]

On the other hand, upon substituting the value of \(a_{2m+1}^2\) from (3.15) into (3.16), it follows that
\[ a_{2m+1} = \tau (m+1)(p_{2m} + h_{2m}) \frac{(m+1)!}{(2\lambda m + 1)!} \cdot \frac{\tau (h_{2m} - p_{2m})}{2(2\lambda m + 1)!} = \frac{\tau h_{2m}}{2\lambda m + 1}. \]

Therefore, we get
\[ |a_{2m+1}| \leq \frac{|\tau|h^{(2m)}(0)|}{(2m)!}. \quad (3.18) \]

So we obtain from (3.17) and (3.18) the desired estimate on the coefficient \(|a_{2m+1}|\) as asserted in (3.2). This completes the proof. \(\square\)

4. Conclusions

If we take
\[ h(z) = p(z) = \left( \frac{1 + z^m}{1 - z^m} \right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots, \]
in Theorem 3.1, we conclude the following result which is an improvement of Theorem 1.2.
Corollary 4.1. Let the function \( f(z) \) given by (1.3) be in the class \( B_{\Sigma m}(\tau, \lambda, \alpha) \). Then

\[
|a_{m+1}| \leq \min \left\{ \frac{2\alpha |\tau|}{\lambda m + 1}, 2\alpha \sqrt{\frac{|\tau|}{(2\lambda m + 1)(m + 1)}} \right\},
\]

and

\[
|a_{2m+1}| \leq \frac{2\alpha^2 |\tau|}{2\lambda m + 1}.
\]

Remark 4.2. It is easy to see, for the coefficient \( |a_{2m+1}| \), that

\[
\frac{2\alpha^2 |\tau|}{2\lambda m + 1} \leq \frac{2\alpha^2 |\tau|^2 (m + 1)}{(\lambda m + 1)^2} + \frac{2\alpha |\tau|}{2\lambda m + 1}.
\]

Thus, clearly, Corollary 4.1 is an improvement of Theorem 1.2.

If we set \( \tau = 1 \) in Corollary 4.1, then the class \( B_{\Sigma m}(\tau, \lambda, \alpha) \) reduces to the class \( A_{\Sigma m}^{\alpha, \lambda} \) which introduced and studied by Sumer Eker [15].

Corollary 4.3. Let the function \( f(z) \) given by (1.3) be in the class \( A_{\Sigma m}^{\alpha, \lambda} \). Then

\[
|a_{m+1}| \leq \begin{cases} \frac{2\alpha \lambda m + 1}{\lambda m + 1}, & \lambda \geq 1 + \sqrt{\frac{m + 1}{m}} \\ \frac{2\alpha}{\sqrt{(2\lambda m + 1)(m + 1)}}, & 1 \leq \lambda < 1 + \sqrt{\frac{m + 1}{m}} \end{cases}
\]

and

\[
|a_{2m+1}| \leq \frac{2\alpha^2}{2\lambda m + 1}.
\]

Remark 4.4. It is easy to see that

\[
\frac{2\alpha}{\lambda m + 1} \leq \frac{2\alpha}{\sqrt{(\lambda m + 1)^2 + \alpha m (1 + 2\lambda m - m\lambda^2)}},
\]

if

\[
\lambda \geq 1 + \sqrt{\frac{m + 1}{m}}
\]

and

\[
\frac{2\alpha}{\sqrt{(2\lambda m + 1)(m + 1)}} \leq \frac{2\alpha}{\sqrt{(\lambda m + 1)^2 + \alpha m (1 + 2\lambda m - m\lambda^2)}}
\]

if

\[
1 \leq \lambda < 1 + \sqrt{\frac{m + 1}{m}}.
\]

On the other hand, for the coefficient \( |a_{2m+1}| \),

\[
\frac{2\alpha^2}{2\lambda m + 1} \leq \frac{2\alpha^2 (m + 1)}{(1 + \lambda m)^2} + \frac{2\alpha}{2\lambda m + 1}.
\]

Thus, clearly Corollary 4.3 provides an improvement of a result which obtained by Sumer Eker [15, Theorem 1].
If we set \( \tau = \lambda = 1 \) in Corollary 4.1, then the class \( \mathcal{B}_{\Sigma_m}(\tau, \lambda, \alpha) \) reduces to the class \( \mathcal{K}_{\Sigma_m}^0 \) which introduced and studied by Srivastava et al. [13].

**Corollary 4.5.** Let the function \( f(z) \) given by (1.3) be in the class \( \mathcal{K}_{\Sigma_m}^0 \). Then

\[
|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(2m+1)(m+1)}}
\]

and

\[
|a_{2m+1}| \leq \frac{2\alpha^2}{2m+1}.
\]

**Remark 4.6.** Corollary 4.5 provides a refinement of a result which obtained by Srivastava et al. [13, Theorem 2].

**Remark 4.7.** If we set \( m = 1 \) in Corollary 4.5, then the class \( \mathcal{K}_{\Sigma_m}^0 \) reduces to the class \( \mathcal{K}_{\alpha} \) which introduced and studied by Srivastava et al. [11].

**Corollary 4.8.** Let the function \( f(z) \) given by (1.1) be in the class \( \mathcal{K}_{\Sigma}^0 \). Then

\[
|a_2| \leq \sqrt{\frac{2}{3}}\alpha,
\]

and

\[
|a_3| \leq \frac{2\alpha^2}{3}.
\]

**Remark 4.9.** Corollary 4.8 provides an improvement of a result which obtained by Srivastava [11, Theorem 1].

For one-fold symmetric bi-univalent functions and for \( \tau = 1 \), the class \( \mathcal{B}_{\Sigma_m}(\tau, \lambda, \alpha) \) reduces to the class \( \mathcal{B}_{\Sigma}(\alpha, \lambda) \) and we obtain the following result which is an improvement of a result which were proven by Frasin and Aouf [5, Theorem 2.2].

**Corollary 4.10.** Let the function \( f(z) \) given by (1.1) be in the class \( \mathcal{B}_{\Sigma}(\alpha, \lambda) \). Then

\[
|a_2| \leq \begin{cases} 
\frac{\sqrt{2}}{2\alpha + 1\lambda}, & 1 \leq \lambda < 1 + \sqrt{2} \\
\frac{\sqrt{2\alpha}}{\lambda + 1\lambda}, & \lambda \geq 1 + \sqrt{2}
\end{cases}
\]

and

\[
|a_3| \leq \frac{2\alpha^2}{2\lambda + 1}.
\]

**Remark 4.11.** Corollary 4.10 provides a refinement of a result which were obtained by Frasin and Aouf [5, Theorem 2.2].

By setting \( h(z) = p(z) = \frac{1 + (1 - 2\beta)z^m}{1 - z^m} = 1 + 2(1 - \beta)z^m + 2(1 - \beta)z^{2m} + \cdots \),

in Theorem 3.1, we deduce the following result.
Corollary 4.12. Let the function \( f(z) \) given by (1.3) be in the class \( \mathcal{B}_{\Sigma_m}^*(\tau, \lambda, \beta) \). Then

\[
|a_{m+1}| \leq \min \left\{ \frac{2(1 - \beta)\tau}{\lambda m + 1}, \sqrt{\frac{4(1 - \beta)|\tau|}{2(2\lambda m + 1)(m + 1)}} \right\},
\]

and

\[
|a_{2m+1}| \leq \frac{2(1 - \beta)|\tau|}{2\lambda m + 1}.
\]

Remark 4.13. It is easy to see, for the coefficient \( |a_{2m+1}| \), that

\[
\frac{2(1 - \beta)|\tau|}{2\lambda m + 1} \leq \frac{2\tau^2(1 - \beta)^2(m + 1)}{(\lambda m + 1)^2} + \frac{2\tau(1 - \beta)^2}{2\lambda m + 1}.
\]

Thus, clearly, Corollary 4.12 is an improvement of Theorem 1.4.

If we set \( \tau = 1 \) in Corollary 4.12, then the class \( \mathcal{B}_{\Sigma_m}^*(\tau, \lambda, \beta) \) reduces to the class \( \mathcal{A}_\lambda^\beta(\beta) \) which introduced and studied by Sumer Eker [15].

Corollary 4.14. Let the function \( f(z) \) given by (1.3) be in the class \( \mathcal{A}_\lambda^\beta(\beta) \). Then

\[
|a_{m+1}| \leq \min \left\{ \frac{2(1 - \beta)}{\lambda m + 1}, \sqrt{\frac{4(1 - \beta)}{2(2\lambda m + 1)(m + 1)}} \right\},
\]

and

\[
|a_{2m+1}| \leq \frac{2(1 - \beta)}{2\lambda m + 1}.
\]

Remark 4.15. It is easy to see that

\[
\frac{2(1 - \beta)}{2\lambda m + 1} \leq \frac{2(1 - \beta)^2(m + 1)}{(1 + \lambda m)^2} + \frac{2(1 - \beta)}{2\lambda m + 1}.
\]

Thus, Corollary 4.14 provides an improvement of a result which obtained by Sumer Eker [15, Theorem 2].

If we take \( \lambda = 1 \) in Corollary 4.14, then the class \( \mathcal{A}_\lambda^\beta(\beta) \) reduces to the class \( \mathcal{H}_{\Sigma_m}^{\beta}(\beta) \) which introduced and studied by Srivastava et al. [13].

Corollary 4.16. Let the function \( f(z) \) given by (1.3) be in the class \( \mathcal{H}_{\Sigma_m}^{\beta}(\beta) \). Then

\[
|a_{m+1}| \leq \min \left\{ \frac{2(1 - \beta)}{m + 1}, \sqrt{\frac{4(1 - \beta)}{(2m + 1)(m + 1)}} \right\},
\]

and

\[
|a_{2m+1}| \leq \frac{2(1 - \beta)}{2m + 1}.
\]

Remark 4.17. Corollary 4.16 provides a refinement of a result which obtained by Srivastava [13, Theorem 3].
If we take $m = 1$ in Corollary 4.16, then the class $\mathcal{H}^\beta_{\Sigma_m}$ reduces to the class $\mathcal{H}^\beta_{\Sigma}$ introduced and studied by Srivastava et al. [11].

**Corollary 4.18.** Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\beta$. Then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}, & 0 \leq \beta \leq \frac{1}{3} \\ \frac{2}{3}(1-\beta), & \frac{1}{3} \leq \beta < 1 \end{cases} \quad (4.5)$$

and

$$|a_3| \leq \frac{2(1-\beta)}{3}.$$ 

**Remark 4.19.** Corollary 4.18 provides a refinement of a result obtained by Srivastava [11, Theorem 2].

For one-fold symmetric bi-univalent functions and for $\tau = 1$, the class $\mathcal{B}^\tau_{\Sigma_m}(\tau, \lambda, \beta)$ reduces to the class $\mathcal{B}_{\Sigma}(\beta, \lambda)$ and we obtain the following result which is an improvement of a result which were proven by Frasin and Aouf [5, Theorem 3.2].

**Corollary 4.20.** Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}(\beta, \lambda)$. Then

$$|a_2| \leq \min \left\{ \frac{2(1-\beta)}{\lambda + 1}, \sqrt{\frac{2(1-\beta)}{2\lambda + 1}} \right\},$$

and

$$|a_3| \leq \frac{2(1-\beta)}{2\lambda + 1}.$$ 

**Remark 4.21.** Corollary 4.20 provides an improvement of a result obtained by Frasin and Aouf [5, Theorem 3.2].

**References**


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