On the Uniform Ergodic for $\alpha$–times Integrated Semigroups

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ABSTRACT: Let $A$ be the generator of an $\alpha$–times integrated semigroup $(S(t))_{t\geq 0}$. We study the uniform ergodicity of $(S(t))_{t\geq 0}$ and we show that the range of $A$ is closed if and only if $\lambda R(\lambda, A)$ is uniformly ergodic. Moreover, we obtain that $(S(t))_{t\geq 0}$ is uniformly ergodic if and only if $\alpha = 0$. Finally, we get that $\frac{1}{t^{\alpha+1}} \int_0^t S(s)ds$ converge uniformly for all $\alpha \geq 0$.

Key Words: Uniformly ergodic, $\alpha$–times integrated semigroup, Generator.

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1. Introduction

Throughout, $X$ denotes a complex Banach space, $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on $X$, let $A$ be a closed linear operator on $X$ with domain $D(A)$, $\ker(A)$, $\mathcal{R}(A)$, $\rho(A)$ and $R(\cdot, A)$, respectively the kernel, the range, the resolvent set and the resolvent function of $A$.

The family $(S(t))_{t\geq 0} \subset \mathcal{B}(X)$ is called $C_0$-semigroup [7] if it has the following properties:

1. The map $t \to S(t)x$ from $[0, +\infty[$ into $X$ is continuous for all $x \in X$,
2. $S(t)S(s)=S(t+s)$,
3. $S(0)=I$.

In this case, its generator $A$ is defined by

$$D(A) = \{x \in X / \lim_{t \to 0^+} \frac{S(t)x - x}{t}\},$$

with

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t}.$$
Now, we recall the notion of $\alpha$-times integrated semigroup which is a generalization of $C_0$-semigroup. Let $\beta \geq -1$ and $f$ be a continuous function. The convolution $j_\beta * f$ is defined for all $t \geq 0$ by

$$j_\beta * f(t) = \begin{cases} \int_0^t \frac{(t-s)^\beta}{\Gamma(\beta+1)} f(s) ds & \text{if } \beta > -1, \\ \int_0^t f(t-s) d\delta_0(s) & \text{if } \beta = -1, \end{cases}$$

where $\Gamma$ is the Euler integral given by $\Gamma(\beta + 1) = \int_0^\infty x^{\beta} e^{-x} dx$, $j_{-1} = \delta_0$ the Dirac measure and for all $\beta > -1$

$$j_\beta : [0, +\infty[ \rightarrow \mathbb{R} \quad t \mapsto \frac{t^\beta}{\Gamma(\beta+1)}.$$

A strongly continuous $(S(t))_{t \geq 0} \subset \mathcal{B}(X)$ is called an $\alpha$-times integrated semigroup where $\alpha > 0$ [3], if $S(0) = 0$ and for all $t, s \geq 0$

$$S_n(t)S_n(s) = \int_t^{t+s} \frac{(s + t - r)^{n-1}}{\Gamma(n)} S_n(r) dr - \int_0^s \frac{(s + t - r)^{n-1}}{\Gamma(n)} S_n(r) dr,$$

where $n - 1 < \alpha \leq n$ and $S_n(t)(x) = (j_{n-\alpha-1} * S)(x)$ for all $x \in X$. By ($\ast$) the following equality hold for all $t, s \geq 0$

$$S(t)S(s) = S(s)S(t).$$

Conversely, let $\alpha \geq 0$ and let $A$ be a linear operator on a Banach space $X$. We recall that $A$ is the generator of an $\alpha$-times integrated semigroup [1] if for some $\omega \in \mathbb{R}$, we have $[\omega, +\infty[ \subseteq \rho(A)$ and there exists a strongly continuous mapping $S : [0, +\infty[ \rightarrow \mathcal{B}(X)$ satisfying

$$\|S(t)\| \leq Me^{\omega t} \quad \text{for all } t \geq 0 \quad \text{and some } M > 0$$

$$R(\lambda, A) = \lambda^\alpha \int_0^{+\infty} e^{-\lambda t} S(t) dt \quad \text{for all } \lambda > \max\{\omega, 0\},$$

in this case, $(S(t))_{t \geq 0}$ is called the $\alpha$-times integrated semigroup and the domain of its generator $A$ is defined by

$$D(A) = \{x \in X : \int_0^t S(s)Ax ds = S(t)x - \frac{t^\alpha x}{\Gamma(\alpha+1)} \}.$$

From the uniqueness Theorem of Laplace Transforms, $(S(t))_{t \geq 0}$ is uniquely determined. In particular, an integrated semigroup is also an 1-times integrated semigroup.

An important example of generators of an $\alpha$-times integrated semigroup is the adjoint $A^*$ on $X^*$ for all $\alpha > 0$, where $A$ is the generator of a $C_0$-semigroup on a Banach space $X$. In particular [3, Examples 3.8], we consider $X = L^1(\mathbb{R})$ and
for all \( f \in D(A) := \{ f \in X : f \text{ is continuous and } f' \in X \} \), we define the linear operator by

\[
Af = -f'.
\]

Since \( X^* = L^\infty(\mathbb{R}) \) and for all \( f \in D(A^*) = \{ f \in X^* : f \text{ continuous and } f' \in X^* \} \), the adjoint \( A^* \) of \( A \) is defined by

\[
A^* f = f'.
\]

Then for all \( \alpha > 0 \), \( A^* \) is a generator of an \( \alpha \)-times integrated semigroup.

An operator \( T \in B(X) \) is called uniformly ergodic if the averages \( \frac{1}{n} \sum_{k=0}^{n-1} T_k \) converge in the uniform operator topology (see [4, Chapitre II]).

In 1974, M. Lin showed in [5, Theorem] (called uniform ergodic theorem), that when \( \lim_{n \to \infty} \frac{T_n}{n} = 0 \), the operator \( T \) is uniformly ergodic if and only if \((1 - T)X \) is closed. In [2] the authors proved that, if \( 1 \) is a pole of the resolvent function, then \( \frac{1}{n} \sum_{k=0}^{n-1} T_k \) converge in norm if and only if \( \lim_{n \to \infty} \frac{T_n}{n} = 0 \).

A semigroup \((S(t))_{t \geq 0} \subset B(X)\) is called uniformly ergodic if \( \frac{1}{t} \int_0^t S(s)ds \) converge uniformly when \( t \to \infty \). Also in [6, Theorem], M. Lin shows for a \( C_0 \)-semigroup \((S(t))_{t \geq 0} \subset B(X)\) satisfying \( \lim_{t \to \infty} \frac{\|S(t)\|}{t} = 0 \), then the following conditions are equivalents:

1. \((S(t))_{t \geq 0} \) is uniformly ergodic,
2. the infinitesimal generator \( A \) has a closed range,
3. \( \frac{1}{n} \sum_{k=0}^{n-1} R(1, A)^k \) converge uniformly,
4. \( \lim_{\lambda \to 0^+} \lambda R(\lambda, A) \) exists in the uniform operator topology.

In this paper, we are motivated by application to the ergodic theory for an \( \alpha \)-times integrated semigroup \((S(t))_{t \geq 0} \) in \( B(X) \) where \( \alpha > 0 \). We prove that when we assume the same conditions of M. Lin’s theorem [6] for an \( \alpha \)-times integrated semigroup \((S(t))_{t \geq 0} \), the integral \( \frac{1}{t} \int_0^t S(s)ds \) converge uniformly if and only if \( \alpha = 0 \).

Moreover, we obtain that if \( \Re(A) \) is closed, then for all \( \alpha \geq 0 \), \( \frac{1}{t^\alpha + 1} \int_0^t S(s)ds \) converge uniformly when \( t \to \infty \).

2. Main results

The next lemma was investigated by W. Arendt [1, Proposition 3.3] in the case of \( n \)-times integrated semigroup, \( n \in \mathbb{N} \). This results has been generalized by M. Heiber [3, Proposition 2.4] to the \( \alpha \)-times integrated semigroup with \( \alpha \geq 0 \) (for interested reader we refer to [8, Lemma 2.1]).
Lemma 2.1. [3, Proposition 2.4] Let $A$ be the generator of an $\alpha$–times integrated semigroup $(S(t))_{t \geq 0} \subset \mathcal{B}(X)$ where $\alpha \geq 0$. Then for all $x \in D(A)$ and all $t \geq 0$ we have

1. $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.
2. $S(t)x = \frac{\alpha^n}{\Gamma(\alpha + 1)} x + \int_0^t S(s)Ax ds$.
3. For all $x \in X$, $\int_0^t S(s)x ds \in D(A)$ and $A \int_0^t S(s)x ds = S(t)x - \frac{\alpha^n}{\Gamma(\alpha + 1)} x$.

Lemma 2.2. Let $A$ be the generator of an $\alpha$–times integrated semigroup $(S(t))_{t \geq 0}$ in $\mathcal{B}(X)$ where $\alpha \geq 0$ satisfying $\lim_{t \to +\infty} \|S(t)t\| = 0$. Then for every $\lambda > \max\{w, 0\}$,

$$\lim_{n \to +\infty} \frac{\|([\lambda R(\lambda, A)]^n)\|}{n} = 0.$$ 

Proof: We put $T(t) := \lambda^\alpha S(t)$, since $\lim_{t \to +\infty} \|S(t)t\| = 0$, then we have

$$\lim_{t \to +\infty} \left\| \frac{T(t)}{t} \right\| = 0.$$

Since

$$R(\lambda, A)x = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)x dt \text{ for all } \lambda > \max\{w, 0\}.$$ 

Then

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x dt.$$ 

From [2, Lemma VIII.1.12], we have

$$R(\lambda, A)^n x = \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t^{n-1}} T(t)x dt \text{ for all } \lambda > \max\{w, 0\}.$$ 

Fix $\epsilon > 0$, then there exists $t_0 \geq 0$ such that for all $t \geq t_0$,

$$\|T(t)\| \leq \epsilon t.$$ 

Since $\|T(t)x\|$ is continuous on $[0, t_0]$, then there exists $K > 0$ such that for all $t \in [0, t_0]$,

$$\|T(t)\| \leq K.$$ 

Therefore

$$\frac{\|([\lambda R(\lambda, A)]^n)\|}{n} \leq \frac{\lambda^n}{n!} K \int_0^{t_0} e^{-\lambda t^{n-1}} dt + \frac{\lambda^n}{n!} \int_{t_0}^\infty e^{-\lambda t^n} dt.$$
Finally as for all \( \lambda > 0 \),

\[
\frac{\lambda^n}{n!} \int_0^\infty e^{-\lambda t} t^n dt = \frac{1}{\lambda},
\]

consequently

\[
\frac{\| [\lambda R(\lambda, A)]^n \|}{n} \leq \frac{K}{n} + \frac{\epsilon}{\lambda}.
\]

Hence we obtain when \( n \to \infty \) and \( \epsilon \to 0 \),

\[
\lim_{n \to +\infty} \frac{\| [\lambda R(\lambda, A)]^n \|}{n} = 0.
\]

Lemma 2.3. Let \( A \) be the generator of an \( \alpha \)-times integrated semigroup \((S(t))_{t \geq 0}\) in \( \mathcal{B}(X) \) where \( \alpha \geq 0 \). Then we have the following assertions:

1. \( \mathcal{R}(A) = \left( \lambda R(\lambda, A) - I \right) X \).
2. \( \text{Ker}(A) = \{ x \in X : S(t)x = \frac{t^\alpha}{\Gamma(\alpha + 1)} x \text{ for all } t \geq 0 \} \)

\( = \{ x \in X : \lambda R(\lambda, A)x = x \} \).

Proof: It is known that for all \( \lambda > \max \{ w, 0 \} \) we have

\[
(\lambda I - A)R(\lambda, A) = I
\]

and for every \( x \in D(A) \)

\[
R(\lambda, A)(\lambda I - A)x = x.
\]

1. Let \( y \in \left( \lambda R(\lambda, A) - I \right) X \), then there exists \( x \in X \) such that

\[
y = \lambda R(\lambda, A)x - x.
\]

Since \( x = (\lambda - A)R(\lambda, A)x \), then

\[
\lambda R(\lambda, A)x - x = AR(\lambda, A)x.
\]

Therefore \( y = AR(\lambda, A)x \in \mathcal{R}(A) \), hence \( \left( \lambda R(\lambda, A) - I \right) X \subset \mathcal{R}(A) \).

Conversely, let \( y \in \mathcal{R}(A) \), then there exists \( x \in D(A) \) such that \( y = Ax \), since

\[
x = R(\lambda, A)(\lambda I - A)x = \lambda R(\lambda, A)x - R(\lambda, A)Ax = \lambda R(\lambda, A)x - R(\lambda, A)y.
\]
Thus
\[ R(\lambda, A)y = \lambda R(\lambda, A) x - x = (\lambda R(\lambda, A) - I)x. \]

Since \((\lambda I - A)\) and \((\lambda R(\lambda, A) - I)\) commute on \(D(A)\), we get
\[ y = (\lambda I - A)R(\lambda, A)y = (\lambda I - A)(\lambda R(\lambda, A) - I)x = (\lambda R(\lambda, A) - I)(\lambda I - A)x = (\lambda R(\lambda, A) - I)z, \]
where \(z = (\lambda I - A)x\), hence \(R(\lambda, A) \subset (\lambda R(\lambda, A) - I)X\).
Then we conclude that \(R(\lambda, A) = (\lambda R(\lambda, A) - I)X\).

2. Firstly, let \(x \in Ker(A)\), then by Lemma 2.1, we obtain
\[ S(t)x = \frac{t^\alpha}{\Gamma(\alpha + 1)}x + \int_0^t S(s)Ax\,ds = \frac{t^\alpha}{\Gamma(\alpha + 1)}x. \]
Hence \(x \in \{x \in X : S(t)x = \frac{t^\alpha}{\Gamma(\alpha + 1)}x \text{ for all } t \geq 0\}\).
Conversely, let \(x \in X\) such that for all \(t \geq 0\)
\[ S(t)x = \frac{t^\alpha}{\Gamma(\alpha + 1)}x. \]
Then by Lemma 2.1, we obtain
\[ A \int_0^t S(s)\,ds = S(t)x - \frac{t^\alpha}{\Gamma(\alpha + 1)}x = 0. \]
Hence for every \(t \geq 0\),
\[ A \int_0^t S(s)\,ds = 0. \]
Thus we conclude that \(\int_0^t S(s)\,ds \in Ker(A)\), hence \(x \in Ker(A)\). Therefore
\[ Ker(A) = \{x \in X : S(t)x = \frac{t^\alpha}{\Gamma(\alpha + 1)}x \text{ for all } t \geq 0\}. \]
Let \(x \in Ker(A)\). Since \(R(\lambda, A)(\lambda - A)x = x\), then
\[ \lambda R(\lambda, A)x = x. \]
Conversely, let $x \in X$ such that $\lambda R(\lambda, A)x = x$, then $x \in D(A)$. Since $(\lambda I - A)R(\lambda, A)x = x$, we deduce that

$$AR(\lambda, A)x = \lambda R(\lambda, A)x - x = 0.$$ 

Hence

$$Ax = A(\lambda R(\lambda, A)x) = \lambda AR(\lambda, A)x = 0.$$ 

Therefore we conclude that $x \in \text{Ker}(A)$ and finally

$$\text{Ker}(A) = \{x \in X : \lambda R(\lambda, A)x = x\}.$$ 

\[\square\]

Now, we give a new characterization of $\text{Ker}(A)$.

Corollary 2.4. Let $A$ be the generator of an $\alpha$–times integrated semigroup $(S(t))_{t \geq 0}$ in $B(X)$ where $\alpha \geq 0$ such that $\lim_{t \to \infty} \frac{\|S(t)\|}{t} = 0$.

If $\alpha \geq 1$, then $A$ is one to one.

Proof: Let $(S(t))_{t \geq 0}$ be an $\alpha$–times integrated semigroup in $B(X)$ where $\alpha \geq 0$ such that $\lim_{t \to \infty} \frac{\|S(t)\|}{t} = 0$. Let $x \in \text{ker}(A)$, then

$$S(t)x = \frac{t^\alpha}{\Gamma(\alpha + 1)}x ; \text{ for all } t \geq 0.$$ 

Therefore we obtain

$$\lim_{t \to \infty} \frac{\|S(t)\|}{t} = \lim_{t \to \infty} \left\| \frac{t^{\alpha-1}}{\Gamma(\alpha + 1)}x \right\| = 0.$$ 

Which means that if $\alpha \geq 1$, then $x = 0$. \[\square\]

Theorem 2.5. Let $A$ be the generator of an $\alpha$–times integrated semigroup $(S(t))_{t \geq 0}$ in $B(X)$ where $\alpha \geq 0$ such that $\lim_{t \to \infty} \frac{\|S(t)\|}{t} = 0$. Then the following conditions are equivalents:

1. $\mathcal{R}(A)$ is closed,

2. $\lambda R(\lambda, A)$ is uniformly ergodic, $\lambda \in \rho(A)$. 

Proof: (1) ⇒ (2) Assume that $R(A)$ is closed, then by Lemma 2.3, we obtain

$$Y = R(A) = (\lambda R(\lambda, A) - I)X.$$  

Hence, by Lemma 2.1, we obtain

$$\lim_{n \to +\infty} \frac{\|\lambda R(\lambda, A)^n\|}{n} = 0.$$  

Therefore, by [5, Theorem], we conclude that $\lambda R(\lambda, A)$ is uniformly ergodic.

(2) ⇒ (1) By the uniform ergodic theorem for the operator $\lambda R(\lambda, A)$, we obtain

$$X = (I - \lambda R(\lambda, A))X \oplus Ker(I - \lambda R(\lambda, A)).$$  

Since $(I - \lambda R(\lambda, A))X$ is closed, then by Lemma 2.3, we deduce that $R(A)$ is closed.

We show in the next proposition that an $\alpha$-times integrated semigroup $(S(t))_{t \geq 0}$ in $\mathcal{B}(X)$ is uniformly ergodic if and only if $\alpha = 0$.

Proposition 2.6. Let $A$ be the generator of an $\alpha$-times integrated semigroup $(S(t))_{t \geq 0} \subset \mathcal{B}(X)$ where $\alpha > 0$ such that $\lim_{t \to \infty} \|S(t)\| = 0$. If $R(A)$ is closed, then $(S(t))_{t \geq 0}$ is not uniformly ergodic.

Proof: Assume that $R(A)$ is closed. Then, by Theorem 2.5 $\lambda R(\lambda, A)$ is uniformly ergodic. So

$$X = (I - \lambda R(\lambda, A))X \oplus \{x \in X : \lambda R(\lambda, A)x = x\}.$$  

Hence by Lemma 2.3, we obtain

$$X = R(A) \oplus Ker(A).$$  

Now, assume that $(S(t))_{t \geq 0}$ is uniformly ergodic and $0 < \alpha < 1$, let $x \in Ker(A)$, then by Lemma 2.2

$$S(t)x = \frac{t^\alpha}{\Gamma(\alpha + 1)}x.$$  

Therefore we obtain

$$\left\| \frac{1}{t} \int_0^t S(s)x ds \right\| = \left\| \frac{1}{t} \int_0^t \frac{s^\alpha}{\Gamma(\alpha + 1)}x ds \right\| = \left\| \frac{t^\alpha}{\Gamma(\alpha + 2)}x \right\|.$$  

Hence $(S(t))_{t \geq 0}$ is not uniformly ergodic.
Let $\alpha \geq 1$. Then by Corollary 2.4, $A$ is one to one. Hence by the ergodic decomposition and Lemma 2.3, we obtain

$$X = \mathcal{R}(A) \oplus \ker(A)$$

Hence $A$ is bijective and $A^{-1}$ is defined for all $X$. Then by the Closed Graph Theorem, we obtain $A^{-1}$ is continuous.

Assume that $(S(t))_{t \geq 0}$ is uniformly ergodic, then there exists an operator $P$ such that $\lim_{t \to \infty} \|t^{-1} \int_0^t S(s) ds - P\| = 0$, $P^2 = P$ and $X = P(X) \oplus \ker(P)$. Thus we conclude that

$$P(X) = \ker(I - \lambda R(\lambda, A)) = \ker(A) = \{0\}.$$ 

Therefore $X = \ker(P) = \mathcal{R}(A)$ and $\lim_{t \to \infty} \|t^{-1} \int_0^t S(s) ds\| = 0$.

For $x \neq 0$, we applied Lemma 2.1, we get

$$A \int_0^t S(s) x ds = S(t)x - \frac{t^\alpha}{\Gamma(\alpha + 1)} x.$$ 

Then

$$\frac{1}{t} A \int_0^t S(s) x ds = \frac{S(t)}{t} x - \frac{t^{\alpha - 1}}{\Gamma(\alpha + 1)} x.$$ 

Since $A$ is invertible, we get

$$\lim_{t \to \infty} \left( \frac{1}{t} A \int_0^t S(s) x ds \right) = A \left( \lim_{t \to \infty} \frac{1}{t} \int_0^t S(s) x ds \right)$$

$$= \lim_{t \to \infty} \frac{S(t)}{t} x - \lim_{t \to \infty} \frac{t^{\alpha - 1}}{t \Gamma(\alpha + 1)} x$$

$$= - \lim_{t \to \infty} \frac{t^{\alpha - 1}}{t \Gamma(\alpha + 1)} x.$$ 

It follows that

$$\lim_{t \to \infty} \frac{t^{\alpha - 1}}{t \Gamma(\alpha + 1)} x = 0,$$

which is absurd because $\alpha \geq 1$ and $x \neq 0$. Finally, we deduce that $(S(t))_{t \geq 0}$ is not uniformly ergodic.

Eventually, we give the following theorem.

**Theorem 2.7.** Let $A$ be the generator of an $\alpha$-times integrated semigroup $(S(t))_{t \geq 0}$ in $\mathcal{B}(X)$ where $\alpha \geq 0$ such that $\lim_{t \to \infty} \left\| \frac{S(t)}{t} \right\| = 0$. If $\mathcal{R}(A)$ is closed, then

$$\frac{1}{t^{\alpha + 1}} \int_0^t S(s) ds$$

converge uniformly for all $\alpha \geq 0$. 

\[\square\]
Proof: Assume that $\mathcal{R}(A)$ is closed and denoted by $Y$.

From Lemma 2.1, we have for all $x \in D(A)$: $AS(t)x = S(t)Ax$, hence $S(t)Y \subseteq Y$.

We denote by $A_1$ the generator of the restriction of $S(t)$ to $Y$, that is the restriction of $A$ to $Y \cap D(A)$. Since $Y = (I - \lambda R(\lambda, A))X$, the uniform ergodic theorem shows that $(I - \lambda R(\lambda, A))$ is invertible on $Y$.

If $A_1y = 0$ for $y \in Y \cap D(A)$, then by $R(\lambda, A)(\lambda I - A)x = x$ for all $x \in D(A)$

we obtain

$$ R(\lambda, A)(\lambda I - A)y = y. $$

Hence

$$ \lambda R(\lambda, A)y = y. $$

Then

$$ y \in \text{Ker}(I - \lambda R(\lambda, A)). $$

That implies $y = 0$, thus $A_1$ is one to one.

Since $(I - \lambda R(\lambda, A))Y \subseteq \mathcal{R}(A_1)$, we conclude that $Y \supseteq \mathcal{R}(A_1) \supseteq (I - \lambda R(\lambda, A))Y = (I - \lambda R(\lambda, A))X = Y = \mathcal{R}(A)$.

Hence $\mathcal{R}(A_1) = Y$, so $A_1^{-1}$ is defined for all $Y$, and by the Closed Graph Theorem, we obtain $A_1^{-1}$ is continuous.

Let $z \in Y$ there is an $x \in Y \cap D(A)$ such that $A_1x = z$ and $\|x\| \leq \|A_1^{-1}\|\|z\|$.

By Lemma 2.1, we have

$$ \int_0^t S(s)A_1xds = S(t)x - \frac{t^\alpha}{\Gamma(\alpha + 1)}x. $$

If $0 \leq \alpha < 1$, we have the ergodic decomposition below

$$ X = Y \oplus \{ x \in X : S(t)x = \frac{t^\alpha}{\Gamma(\alpha + 1)}x ; t \geq 0 \}. $$

Therefore we obtain

$$ \left\| \frac{1}{t^{\alpha + 1}} \int_0^t S(s)zds \right\| = \left\| \frac{1}{t^{\alpha + 1}} \int_0^t S(s)A_1xds \right\| $$

$$ = \left\| \frac{1}{t^{\alpha + 1}} \left( S(t)x - \frac{t^\alpha}{\Gamma(\alpha + 1)}x \right) \right\| $$

$$ \leq \left\| \frac{S(t)}{t^{\alpha + 1}}x \right\| + \left\| \frac{I}{t\Gamma(\alpha + 1)}x \right\| $$

$$ \leq \left( \left\| \frac{S(t)}{t^{\alpha + 1}} \right\| + \left\| \frac{I}{t\Gamma(\alpha + 1)} \right\| \right) \|x\| $$

$$ \leq \left( \left\| \frac{S(t)}{t^{\alpha + 1}} \right\| + \left\| \frac{I}{t\Gamma(\alpha + 1)} \right\| \right) \|A_1^{-1}\|\|z\|. $$
For $t \to \infty$, we obtain the uniform convergence to 0 on $Y$.

Now, let $z \in \{x \in X : S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x$ for all $t \geq 0\}$. Therefore we obtain

$$\left\| \frac{1}{t^{\alpha+1}} \int_0^t S(s)z ds \right\| = \left\| \frac{1}{t^{\alpha+1}} \int_0^t \frac{t^\alpha}{\Gamma(\alpha+1)}z ds \right\| = \left\| \frac{1}{t^{\alpha+1}} \left[ \frac{t^\alpha}{(\alpha+1)\Gamma(\alpha+1)}z \right]_0^t \right\| = \left\| \frac{I}{\Gamma(\alpha+2)}z \right\|.$$

Hence, we get the convergence to $\frac{z}{\Gamma(\alpha+2)}$ on $\{x \in X : S(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x; t \geq 0\}$. By the ergodic decomposition above, we conclude that $\frac{1}{t^{\alpha+1}} \int_0^t S(s)ds$ converge uniformly for $0 \leq \alpha < 1$.

If $\alpha \geq 1$, we find that $\text{Ker}(A) = \{0\}$. By the ergodic decomposition and by Lemma 2.3,

$$X = \mathcal{R}(A) \oplus \text{Ker}(A) = \mathcal{R}(A).$$

Hence $A^{-1}$ is defined for all $X$ and by the Closed Graph Theorem, we obtain $A^{-1}$ is continuous. Then for $z \in X$ there exists $x \in D(A)$ such that $Ax = z$ and $\|x\| \leq \|A^{-1}\| \|z\|$. Hence

$$\left\| \frac{1}{t^{\alpha+1}} \int_0^t S(s)z ds \right\| = \left\| \frac{1}{t^{\alpha+1}} \int_0^t S(s)Ax ds \right\| = \left\| \frac{1}{t^{\alpha+1}} \left( S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x \right) \right\| \leq \left\| \frac{S(t)}{\alpha+1}x \right\| + \left\| \frac{I}{t \Gamma(\alpha+1)}x \right\| \leq \left( \left\| \frac{S(t)}{\alpha+1} \right\| + \left\| \frac{I}{t \Gamma(\alpha+1)} \right\| \right) \|A^{-1}\| \|z\|.$$

For $t \to \infty$, we obtain the uniform convergence to 0 on $X$.

**Remark 2.8.** As mentioned above, the uniform ergodicity implies the ergodic decomposition of $X = \text{Ker}(A) \oplus \mathcal{R}(A)$. But the convergence obtained in the last theorem does not imply this decomposition, which is means that the converse of implication above that is no satisfy in general.

**Acknowledgments**

The authors are thankful to the referee for his valuable comments and suggestions.
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