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Chain conditions in modular lattices with applications to Grothendieck categories and torsion theories

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Contents

Preface

This text is an expanded version of the Lecture Notes [15] of a minicourse that has been given by the author at the XXIII *Escola de Álgebra,* Universidade Estadual de Maringá, July 27-August 1, 2014, Maringá, Paraná, Brazil. The minicourse consisted of five 50-minute lectures, and each chapter of this text corresponds to a lecture.

The text presents in a compact way some basics of Lattice Theory with a great emphasis on chain conditions in modular lattices, that are then applied to Grothendieck categories and module categories equipped with hereditary torsion theories to obtain immediately and in a unified manner significant results in these areas. We also include other results of Algebraic Theory of Lattices that are interesting in their own right.

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Bucharest, January 31, 2015 Toma Albu

General notation

$$
\begin{aligned}
\mathbb{N} &= \{1, 2, \ldots\},\ \text{ the set of all natural numbers} \\
\mathbb{Z} &= \text{the ring of all rational integers} \\
\mathbb{Q} &= \text{the field of all rational numbers} \\
\mathbb{R} &= \text{the field of all real numbers} \\
\mathbb{C} &= \text{the field of all complex numbers} \\
|M| &= \mathrm{Card}(M),\ \text{ the cardinal number of an arbitrary set } M \\
1_M &= \text{the identity map on the set } M \\
\mathbf{Ab} &= \text{the category of all Abelian groups} \\
\text{Mod-}R &= \text{the category of all unital right modules over a unital ring } R \\
M_R &= M \text{ is a right } R\text{-module}
\end{aligned}
$$

Introduction

The main purpose of this text is to present some topics of *Lattice Theory,* with a great emphasis on chain conditions in modular lattices, that have nice applications to important results of *Ring and Module Theory,* including their relativization and absolutization. Specifically, we illustrate a ***general strategy*** which consists on putting a *module-theoretical* result into a *latticial frame* (we call it *latticization*), in order to translate that result to *Grothendieck categories* (we call it *absolutization*) and *module categories* equipped with *hereditary torsion theories* (we call it *relativization*).

More precisely, if $\mathbb{P}$ is a problem, involving subobjects or submodules, to be investigated in Grothendieck categories or in module categories with respect to hereditary torsion theories, our strategy consists of the following three steps:

I. *Translate/formulate,* if possible, the problem $\mathbb{P}$ into a *latticial setting.*
II. *Investigate* the obtained problem $\mathbb{P}$ in this *latticial frame.*
III. *Back to basics,* i.e., to Grothendieck categories and module categories equipped with hereditary torsion theories.

This approach is very natural and simple, because we ignore the specific context of Grothendieck categories and module categories equipped with hereditary torsion theories, focusing only on those latticial properties which are relevant to our given specific categorical or relative module-theoretical problem $\mathbb{P}$. The renowned *Hopkins-Levitzki Theorem* and *Osofsky-Smith Theorem* from Ring and Module Theory are among the most relevant illustrations of the power of this strategy.

Each of the five chapters comprising this volume is focused on a single topic. The first two chapters are devoted to stating and proving several basic results of Algebraic Theory of Lattices having their roots in Module Theory. These results are then applied in the last two chapters to Grothendieck categories and module categories equipped with hereditary torsion theories to obtain immediately and in a unified manner the categorical and relative counterparts of the two renowned theorems of Ring and Module Theory mentioned above. The middle Chapter 3 presents in a compact way, mostly without proofs, the basics of Abelian categories and hereditary torsion theories, including quotient categories, Grothendieck categories, and the renowned Gabriel-Popescu Theorem. We shall now describe in more details the contents of each of these five chapters.

Chapter 1 presents, with complete proofs, some basic notions, terminology, notation, and results on lattices that will be used throughout the text. Special attention was given to the concepts of modular lattice, complemented lattice, upper continuous lattice, complement element, essential element, closed element, pseudo-complement element, E-complemented lattice, pseudo-complemented lattice, strongly pseudo-complemented lattice, and essentially closed lattice. Of course, all the results in this chapter have "duals" obtained by using the opposite lattice. No further proofs are required for them.

In addition, results can be obtained for modules by applying the above results to the lattice of all submodules of a module. It should be noted that the new concepts of completely E-complemented lattice and strongly pseudo-complemented lattice introduced in [34] appear for the first time in a volume. Many results of this chapter are not only used in the last two chapters for their immediate applications to Grothendieck categories and torsion theories, but are also interesting in their own right.

In Chapter 2 we study chain conditions in modular lattices. Specifically, we discuss Noetherian lattices, Artinian lattices, lattices with finite length, Goldie dimension of lattices, and Krull dimension, dual Krull dimension, classical Krull dimension, and Gabriel dimension of arbitrary posets. A cornerstone in the development of modern Ring Theory is the concept of Goldie dimension. Modular lattices provide a very natural setting for the development of this dimension and therefore we prove thoroughly in this chapter the basic properties of Goldie dimension of arbitrary modular lattices, which are not only used in the next chapters, but have also intrinsic value. We also discuss the dual Goldie dimension a lattice L as being the Goldie dimension of its opposite lattice L^o and obtain at once results on it just by translating, without requiring any proofs, the results on the Goldie dimension of L into L^o. In addition, results can be instantly obtained for the dual Goldie dimension of modules simply by applying the above latticial results to the lattice of all submodules of a module.

Chapter 3 is a preparation for the last two chapters of this text, where Grothendieck categories and hereditary torsion theories will show up when applying the latticial results from Chapters 1 and 2 to these concrete cases. We first present in a very compact manner, but without proofs, all the basic concepts of Category Theory: direct sum, direct product, subobject, quotient object, additive category, kernel, cokernel, image, coimage, Abelian category, quotient category, Grothendieck's axioms AB1, AB2, AB3, AB4, and AB5, leading to the definition of a Grothendieck category. A method for obtaining new Grothendieck categories from the known ones is the *localization* technique: for any localizing subcategory $\mathcal{T}$ of the category Mod-R of all unital right modules over a unital ring R, the quotient category Mod-$R/\mathcal{T}$ is a Grothendieck category. We present and explain then the statement of the renowned *Gabriel-Popescu Theorem*, which roughly says that all Grothendieck categories are obtained in this way, up to a category equivalence. We then discuss the concept of a hereditary torsion theory, focusing on the lattice $\mathrm{Sat}_\tau(M_R)$ of all τ-saturated submodules of a right module M_R, where $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on Mod-R. It turns out that this lattice is isomorphic to the lattice of all subobjects of the object $T_\tau(M)$ in the Grothendieck category Mod-$R/\mathcal{T}$, where $T_\tau : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$ is the canonical functor.

Chapter 4 presents various aspects of the celebrated *Hopkins-Levitzki Theorem*, abbreviated H-LT, discovered independently in 1939, so 75 years ago, by C. Hopkins and J. Levitzki. This theorem states that any right Artinian ring with identity is right Noetherian, or, equivalently, any Artinian right R-module over a right Artinian ring R is Noetherian. In the last fifty years, especially in the 1970's, 1980's, and 1990's this result has been generalized to modules relative to a hereditary torsion theory (Relative H-LT), to Grothendieck categories (Absolute or Categorical H-LT), and to arbitrary modular lattices (Latticial H-LT).

The aim of this chapter is to briefly explain all these aspects of the H-LT, their dual formulations, the connections between them, as well as to present other newer aspects of it involving the concepts of Krull and dual Krull dimension. Let us mention that the only two module-theoretical proofs available in the literature of the Relative H-LT, due to Miller and Teply [66] and Faith [49], are very long and complicated. We show in a unified manner that this result, as well as the Absolute H-LT, are immediate consequences of the Latticial H-LT, whose proof is very short and simple, illustrating thus the power of our main strategy explained above.

Chapter 5 is devoted to another famous theorem in Module Theory, namely the *Osofsky-Smith Theorem,* abbreviated O-ST, giving sufficient conditions for a finitely generated or cyclic module to be a finite direct sum of uniform submodules. More precisely, it says that a finitely generated (respectively, cyclic) right R-module such that all of its finitely generated (respectively, cyclic) subfactors are CS modules is a finite direct sum of uniform submodules. We first present the proof (in fact, only a sketch of it because of space limitation of this text) of the latticial counterpart of this theorem, and then apply it to derive immediately the Categorical (or Absolute) O-ST and the Relative O-ST. We believe that the reader will be once more convinced of the power of our strategy when extending some important results of Module Theory to Grothendieck categories and to module categories equipped with hereditary torsion theories by passing first through their latticial counterparts.

The statement and proof of the Categorical O-ST offer a good example showing that assertions like "*basically the same proof for modules works in the categorical setting*" are very risky and may lead to incorrect results; in other words, not all module-theoretical proofs can be easily transferred into a categorical setting just by saying that they can be done mutatis-mutandis. Indeed, as we shall see in the last part of Section 5.3, some well-hidden errors in statements/results occurring in the literature on the Categorical O-ST could be spotted only by using our latticial approach of it. So, we do not only correctly absolutize the module-theoretical O-ST but also provide a correct proof of its categorical extension by passing first through its latticial counterpart.

An effort was made to keep the account as self-contained as possible. However, a certain knowledge of Ring Theory, Module Theory, and Category Theory, at the level of a graduate course is needed for a good understanding of most part of this text. Thus, we assume from the reader some familiarity with basic notions and facts on rings, modules, and categories as presented e.g., in the books of Anderson and Fuller [39], Lam [58] and/or Wisbauer [88], although, whenever it was possible, some of them were included in the text. When proofs are not included, detailed information on where they can be found is given.

CHAPTER 1

LATTICE BACKGROUND

This first chapter presents some basic notions, terminology, notation, and results on lattices that will be used throughout the text. A special attention was given to the concepts of modular lattice, complemented lattice, upper continuous lattice, essential element, closed element, pseudo-complement element, E-complemented lattice, pseudo-complemented lattice, strongly pseudo-complemented lattice, and essentially closed lattice. Their dual counterparts in opposite lattices are also discussed.

For all undefined notation and terminology on lattices, as well as for more results on them, the reader is referred to [41], [42], [44], [56], and/or [85].

1.1. Basic concepts

In this section we present some basic and well-known material about lattices. In particular, we highlight some facts about modular lattices that will be used repeatedly in the following sections.

Posets

Recall that a *partially ordered set* (more briefly, a *poset*) is a pair $(P, \leqslant)$ consisting of a non-empty set P and a binary relation $\leqslant$ on P which is reflexive, anti-symmetric, and transitive, i.e., satisfies the following three conditions:

(i) $a \leqslant a$ for all $a \in P$;
(ii) given $a, b \in P$, $a \leqslant b$ and $b \leqslant a$ together imply $a = b$;
(iii) given $a, b, c \in P$, $a \leqslant b$ and $b \leqslant c$ together imply $a \leqslant c$.

Very often, a poset $(P, \leqslant)$ will be denoted shortly by P.

Another partial order $\leqslant^o$ can be defined on P as follows: given $a, b \in P$, $a \leqslant^o b$ if and only if $b \leqslant a$. The poset $(P, \leqslant^o)$ is called the *opposite* or *dual* poset of $(P, \leqslant)$ and will be denoted by P^o. The poset P is called *trivial* provided $a = b$ for all elements a and b in P such that $a \leqslant b$.

Let S be a subset of a poset P. An *upper bound* for S in P is an element $u \in P$ such that $s \leqslant u$ for all $s \in S$. An element $s_0 \in S$ is a *greatest* or *last* element of S if $s \leqslant s_0$ for all $s \in S$; there can be at most one greatest element of S. Similarly one define *lower bound* and *least element*. A *least upper bound* is a least element in the set of upper bounds for S, and similarly for *greatest lower bound* for S.

For a poset P and elements a and b in P such that $a \leqslant b$ we set

$$b/a := [a, b] = \{\, x \in P \mid a \leqslant x \leqslant b \,\},$$

An *initial interval* (respectively, a *quotient interval*) of b/a is any interval c/a (respectively, b/c) for some $c \in b/a$. A *subfactor* of P is any interval b/a with $a \leqslant b$.

For any $a \in P$, we also set

$$[a) := \{ x \in P \mid a \leqslant x \},$$
$$(a] := \{ x \in P \mid x \leqslant a \}.$$

If the poset P has a least element, which is usually denoted by 0, then $(a] = a/0$ for every element a in P, and if P has a greatest element, which is usually denoted by 1, then $[a) = 1/a$ for every $a \in P$. Clearly a poset P has a least element if and only if its opposite poset P^o has a greatest element. We say that the poset P is *bounded* if P has both a least element and a greatest element.

As one might expect, the *zero poset* is the poset, we shall denote by 0, consisting of a single element also denoted by 0. A poset P will be called *non-zero* if $P \neq 0$. A *proper element* of a poset P with greatest element 1 is any element $a \in P$ with $a \neq 1$.

Our main aim throughout this text is to investigate chains of elements in posets and, in particular, in lattices. A non-empty subset C of a poset P is said to be a *chain* (also called *totally ordered* or *linearly ordered*) provided for any elements x and y in C either $x \leqslant y$ or $y \leqslant x$.

A poset P is said to be *Noetherian* if there is no sequence $x_1, x_2, x_3, \ldots$ of elements of P such that $x_1 < x_2 < x_3 < \ldots$, and P is said to be *Artinian* if there is no sequence $x_1, x_2, x_3, \ldots$ of elements of P such that $x_1 > x_2 > x_3 > \ldots$, i.e., if the opposite poset P^o is Noetherian. Equivalently, a poset P is Noetherian (respectively, Artinian) if, for every sequence $x_1, x_2, x_3, \ldots$ of elements in P with

$$x_1 \leqslant x_2 \leqslant x_3 \leqslant \ldots \text{ (respectively, } x_1 \geqslant x_2 \geqslant x_3 \geqslant \ldots \text{)}$$

there exists a positive integer n such that $x_n = x_{n+1} = \ldots$.

The sequences in P considered above are also called *ascending chains* (respectively, *descending chains*), and so, we say that a Noetherian (respectively, Artinian) poset satisfies the *ascending chain condition* or, more simply, the *ACC* (respectively, the *descending chain condition* or, more simply, the *DCC*).

The concepts of Noetherian and Artinian posets can be also equivalently reformulated in terms of maximal and minimal elements. By a *maximal element* of a non-empty subset S of a poset P we mean an element $m \in S$ such that whenever $m \leqslant x$ and $x \in S$ then $x = m$. Similarly, n is a *minimal element* of S if $n \in S$ and whenever $y \leqslant n$ and $y \in S$ then $y = n$.

Not every non-empty subset of a poset need have either a maximal element or a minimal element. Moreover, if a non-empty set S has a maximal element then it can happen that S has an infinite number of maximal elements, and similarly for minimal elements. It is not hard to think of examples. If P has a greatest element 1, then 1 is a maximal element of P, and if P has a least element 0, then 0 is a minimal element of P. Clearly, any finite poset has maximal elements and minimal elements. The existence of maximal elements in infinite posets is assured by the following well-known result:

Zorn's Lemma. *Let P be a poset such that every chain of P has an upper bound in P. Then P has at least a maximal element.* □

The next result presents the characterization of Noetherian (respectively, Artinian) posets using maximal (respectively, minimal) elements.

Proposition 1.1.1. *A poset P is Noetherian* (*respectively, Artinian*) *if and only if every non-empty subset of P has a maximal* (*respectively, minimal*) *element.*

Proof. Suppose that P is not Noetherian. Then there exists an ascending chain $a_1 < a_2 < \dots$ of elements of P. Let $X = \{ a_i \mid i \in \mathbb{N} \}$. Recall that $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes throughout this text the set of all natural numbers. Clearly, the subset X of P is non-empty and contains no maximal element.

Conversely, suppose that there exists a non-empty subset Y of P such that Y does not contain a maximal element. Let $y_1 \in Y$. Because y_1 is not a maximal element of Y, there exists $y_2 \in Y$ such that $y_1 < y_2$. Next, because y_2 is not a maximal element in Y, there exists $y_3 \in Y$ such that $y_2 < y_3$. In this way, the set Y contains an infinite ascending chain $y_1 < y_2 < \dots$ and it follows that P is not Noetherian. To prove the Artinian result use the opposite poset P^o. □

Lattices

By a *lattice* we mean a poset $(L, \leqslant)$ such that every pair of elements a, b in L has a *greatest lower bound* or *infimum* $a \wedge b$ (also called *meet*) and a *least upper bound* $a \vee b$ or *supremum* (also called *join*), i.e.,

(i) $a \wedge b \leqslant a$, $a \wedge b \leqslant b$, and $c \leqslant a \wedge b$ for all $c \in L$ with $c \leqslant a$, $c \leqslant b$;

(ii) $a \leqslant a \vee b$, $b \leqslant a \vee b$, and $a \vee b \leqslant d$ for all $d \in L$ with $a \leqslant d$, $b \leqslant d$.

Note that, for given $a, b \in L$, $a \wedge b$ and $a \vee b$ are unique, and

$$a \leqslant b \iff a = a \wedge b \iff b = a \vee b.$$

It is easy to deduce that if a, b, c are elements of L such that $a \leqslant b$ then $a \wedge c \leqslant b \wedge c$ and $a \vee c \leqslant b \vee c$. Note further that the opposite poset L^o is also a lattice and that, given $a, b \in L$, the greatest lower bound of a and b in L^o is the least upper bound $a \vee b$ of a and b in L and the least upper bound of a and b in L^o is the greatest lower bound $a \wedge b$ of a and b in L. Clearly L is the opposite lattice of the lattice L^o.

A non-empty subset K of L is said to be a *sublattice* of L provided $x \wedge y \in K$ and $x \vee y \in K$ whenever $x \in K$ and $y \in K$.

Let $(L, \leqslant, \wedge, \vee)$ (or, more simply, L) be any lattice. It is easy to check that

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \text{ and } (a \vee b) \vee c = a \vee (b \vee c),$$

for all elements a, b, c in L. Using this fact and induction it is easy to prove that if $S = \{a_1, \dots, a_n\}$ is any finite subset of L then there exists a unique element b of L such that $b \leqslant a_i \, (1 \leqslant i \leqslant n)$ and $b \geqslant x$ for every $x \in L$ such that $x \leqslant a_i \, (1 \leqslant i \leqslant n)$. The element b is thus the *greatest lower bound* of the set S and is denoted by $\bigwedge S := \bigwedge_{s \in S} s$ or by $a_1 \wedge \dots \wedge a_n$. Similarly, the set S has a least upper bound which will be denoted by $\bigvee S := \bigvee_{s \in S} s$ or by $a_1 \vee \dots \vee a_n$.

Let L and L' be lattices. A mapping $f : L \longrightarrow L'$ is called a (*lattice*) *morphism* if

$$f(a \wedge b) = f(a) \wedge f(b) \quad \text{and} \quad f(a \vee b) = f(a) \vee f(b),$$

for all $a, b \in L$. If, in addition, f is a bijection then f is called an *isomorphism* and we write $L \simeq L'$.

Proposition 1.1.2. *Let L and L' be two lattices. Then, the following statements are equivalent for a mapping $f : L \longrightarrow L'$.*

(1) *f is an isomorphism.*

(2) *f is a bijection and for any $a, b \in L$, ($f(a) \leqslant f(b)$ in $L' \iff a \leqslant b$ in L).*

Proof. (1) $\Longrightarrow$ (2) Suppose first that $f : L \longrightarrow L'$ is an isomorphism. Then f is a bijection. Let $a \leqslant b$ in L. Then $f(a) = f(a \wedge b) = f(a) \wedge f(b) \leqslant f(b)$. Now suppose that a and b are elements of L with $f(a) \leqslant f(b)$. Then $f(a) = f(a) \wedge f(b) = f(a \wedge b)$ so that $a = a \wedge b \leqslant b$.

(2) $\Longrightarrow$ (1) Conversely, suppose that f is a bijection with the stated property. Let $a, b \in L$. Then $a \leqslant a \vee b$ gives that $f(a) \leqslant f(a \vee b)$. Similarly, $f(b) \leqslant f(a \vee b)$. Thus $f(a) \vee f(b) \leqslant f(a \vee b)$. There exists $c \in L$ such that $f(c) = f(a) \vee f(b)$. Note that $f(a) \leqslant f(c)$ so that, by hypothesis, $a \leqslant c$. Similarly $b \leqslant c$ and hence $a \vee b \leqslant c$. It follows that $f(a \vee b) \leqslant f(c) = f(a) \vee f(b)$. Thus $f(a \vee b) = f(a) \vee f(b)$. Similarly, $f(a \wedge b) = f(a) \wedge f(b)$. Thus f is an isomorphism. $\square$

Note that Proposition 1.1.2 and its proof show that a bijection f from a lattice L to a lattice L' such that $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in L$ is an isomorphism. Note also that in Proposition 1.1.2 it is not sufficient to suppose that f is a bijection such that $f(a) \leqslant f(b)$ for all $a \leqslant b$ in L. For example, let L denote the lattice of all positive divisors of 6 with the ordering given by divisibility and let L' denote the lattice $\{1, 2, 3, 4\}$ with the usual ordering. Consider the mapping $f : L \longrightarrow L'$ defined by $f(6) := 4$ and $f(i) := i$ for all $i \in L \setminus \{6\}$. Then $f : L \longrightarrow L'$ is a bijection such that $f(i) \leqslant f(j)$ for all $i \leqslant j$ in L but f is not an isomorphism, because $f(2) \leqslant f(3)$ in L' but $2 \nmid 3$ in L.

We are now going to reformulate Proposition 1.1.2 in the language of *Category Theory* (see Section 3.1). If we denote by **Pos** the class of all posets and by **Lat** the class of all lattices, then they can be considered as categories, where the morphisms in **Pos** are the increasing (or order-preserving) mappings, and the morphisms in **Lat** are the mappings defined just before Proposition 1.1.2. Recall that a mapping $f : P \longrightarrow P'$ between the posets P and P' is said to be *increasing* or *order-preserving* if $f(x) \leqslant f(y)$ in P' for all $x \leqslant y$ in P.

Notice that statement (2) in Proposition 1.1.2 means exactly that the mapping $f : L \longrightarrow L'$ is a bijection such that f and its inverse f^{-1} are increasing mappings, i.e., f is an isomorphism in the category **Pos**. Thus, Proposition 1.1.2 can be reformulated as follows: *a mapping $f : L \longrightarrow L'$ between two lattices is an isomorphism in the category* **Lat** *if and only if f is an isomorphism in the category* **Pos**.

Modular lattices

Now we come to the central idea in this text. A lattice L is called *modular* provided

$$a \wedge (b \vee c) = b \vee (a \wedge c),$$

for all a, b, c in L with $b \leqslant a$. Clearly every sublattice of a modular lattice is also modular.

Let A be an Abelian group. Then the set $\mathcal{L}(A)$ of all subgroups of A is a poset with respect to the partial order given by the usual inclusion, and even a lattice when we define for given subgroups B and C of A,

$$B \wedge C := B \cap C \quad \text{and} \quad B \vee C := B + C.$$

Note that $\mathcal{L}(A)$ has least element the zero subgroup 0 and greatest element A. Moreover, it is easy to check that if B, C, D are subgroups of A such that $C \subseteq B$, then

$$B \cap (C + D) = C + (B \cap D),$$

so that $\mathcal{L}(A)$ is a modular lattice, and hence so too is every sublattice of $\mathcal{L}(A)$. In particular, let R be a unital ring and let M be a right R-module. Then R is an Abelian group and hence the set of all two-sided ideals of R and the set of all right ideals of R are both modular lattices. In the same way, because M is an Abelian group, the set of all submodules of M is a modular lattice. Note the following elementary fact.

PROPOSITION 1.1.3. *Let L be any modular lattice. Then the opposite lattice L^o is also modular.*

PROOF. Let a, b, c be any elements of L with $b \leqslant a$. Then

$$a \wedge (b \vee c) = b \vee (a \wedge c).$$

Note that $a \leqslant^o b$ and $b \vee (a \wedge c) = a \wedge (b \vee c)$. It follows that L^o is a modular lattice. □

The next result gives a characterization for a lattice to be modular and we shall need this characterization later.

PROPOSITION 1.1.4. *A lattice L is modular if and only if, for all $a, b \in L$, the mapping*

$$\varphi : (a \vee b)/a \longrightarrow b/(a \wedge b),\ \varphi(x) = x \wedge b,\ \forall x \in (a \vee b)/a,$$

is a lattice isomorphism with inverse

$$\psi : b/(a \wedge b) \longrightarrow (a \vee b)/a,\ \psi(y) = y \vee a,\ \forall y \in b/(a \wedge b).$$

PROOF. Suppose first that L is modular. Let $x \in (a \vee b)/a$. Then

$$\varphi(x) = x \wedge b \in b/(a \wedge b)$$

and

$$(\psi \circ \varphi)(x) = a \vee (x \wedge b) = x \wedge (a \vee b) = x.$$

Thus $\psi \circ \varphi = 1_{(a \vee b)/a}$.

Also, for any $y \in b/(a \wedge b)$, $\psi(y) = y \vee a \in (a \vee b)/a$ and

$$(\varphi \circ \psi)(y) = b \wedge (y \vee a) = y \vee (b \wedge a) = y.$$

Thus $\varphi \circ \psi = 1_{b/(a \wedge b)}$. It follows that φ is a bijection.

Moreover, if $x \leqslant x'$ in $(a \vee b)/a$ then $\varphi(x) = x \wedge b \leqslant x' \vee b = \varphi(x')$. In addition, if $y \leqslant y'$ in $b/(a \wedge b)$ then $\psi(y) = y \vee a \leqslant y' \vee a = \psi(y')$. By Proposition 1.1.2, φ is a lattice isomorphism.

Conversely, suppose that φ satisfies the stated condition. Let $a, b, c \in L$ with $a \leqslant c$. Then

$$c \wedge (a \vee b) = (\psi \circ \varphi)(c \wedge (a \vee b)) = \psi(c \wedge (a \vee b) \wedge b) = \psi(c \wedge b) = a \vee (c \wedge b).$$

It follows that L is modular. □

We denote by $\mathcal{L}$ (respectively, $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_{0,1}$) the class of all lattices (respectively, lattices with least element 0, lattices with greatest element 1, lattices with least element 0 and greatest element 1), and L will always designate a member of $\mathcal{L}$. In addition, we shall denote by $\mathcal{M}$ the class of all modular lattices. The notation $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_{0,1}$ have similar meanings.

Throughout this text we shall assume that **all lattices are modular**. In fact, some results do not require this hypothesis but it will be left to the reader to spot these. Modularity will be used repeatedly without further comment.

Complemented lattices

If L is a lattice with least element 0 and greatest element 1, then an element $c \in L$ is a *complement* (in L) if there exists an element $a \in L$ such that $a \wedge c = 0$ and $a \vee c = 1$; we say in this case that c is a *complement* of a (in L). For example, 1 is a complement of 0 and 0 is a complement of 1. One denotes by $D(L)$ the set of all complements of L, so $\{0,1\} \subseteq D(L)$. The lattice L is called *indecomposable* if $L \neq \{0\}$ and $D(L) = \{0,1\}$. An element $a \in L$ is said to be an *indecomposable* element if $a/0$ is an indecomposable lattice.

However, a given element a need not have a complement. The lattice L is called *complemented* if it has least element and greatest element, and every element has a complement. For example, if R is an arbitrary unital ring and M a right unital R-module, then the lattice $\mathcal{L}(M)$ of all submodules of M is complemented (respectively, indecomposable) if and only if the module M is semisimple (respectively, indecomposable).

Proposition 1.1.5. *Let $a \leqslant b$ be elements of a complemented* (*modular*) *lattice L. Then the sublattice b/a of L is also complemented.*

Proof. Let $c \in L$ with $a \leqslant c \leqslant b$. There exists $d \in L$ such that $1 = c \vee d$ and $c \wedge d = 0$. Then $b = b \wedge (c \vee d) = c \vee (b \wedge d) = c \vee f$, where $f := a \vee (b \wedge d) \in b/a$. Moreover, $c \wedge f = c \wedge (a \vee (b \wedge d)) = a \vee (c \wedge b \wedge d) = a \vee (0 \wedge b) = a$. It follows that b/a is complemented. $\square$

Let us illustrate these ideas by an elementary very useful example.

Example 1.1.6. Let F be any field, let V be any infinite dimensional vector space over F, and let H denote the lattice of all subspaces of V. Then the set G of all finite dimensional subspaces of V is a sublattice of H. The lattice H has least element the zero subspace and greatest element V and is complemented. The sublattice G has least element the zero subspace and no greatest element. $\square$

Not every complemented lattice is modular. The set $\{1, 3, 4, 6, 12\}$ can be made into a lattice L where the ordering is given by divisibility. Thus L has greatest element 12 and least element 1. It is easy to check that L is complemented but is not modular.

Complete lattices

A poset in which every subset S has a least upper bound $\bigvee_{s \in S} s$ (also denoted by $\bigvee S$ or by $\sup(S)$) and a greatest lower bound $\bigwedge_{x \in S} x$ (also denoted by $\bigwedge S$ or by $\inf(S)$) is called a *complete lattice.* In a complete lattice L there exist a greatest element $\sup(L)$, denoted by 1, and a least element $\inf(L)$, denoted by 0. Observe that $0 = \sup(\varnothing)$ and $1 = \inf(\varnothing)$.

For example, in the lattice $\mathcal{L}(M)$ of all submodules of a module M, for any subset $\mathcal{X} \subseteq \mathcal{L}(M)$ clearly we have

$$\bigvee \mathcal{X} = \sum_{B \in \mathcal{X}} B \quad \text{and} \quad \bigwedge \mathcal{X} = \bigcap_{B \in \mathcal{X}} B,$$

so that $\mathcal{L}(M)$ is a complete lattice. The lattice G of Example 1.1.6 does not have a greatest element and so is not complete, but every non-empty subset has a greatest lower bound.

Note the following two elementary results that will be used in Chapter 2.

LEMMA 1.1.7. *Let $S_i\,(1 \leqslant i \leqslant n)$ be subsets of a complete lattice L. Then*

$$\bigvee(S_1 \cup \cdots \cup S_n) = \big(\bigvee S_1\big) \vee \cdots \vee \big(\bigvee S_n\big).$$

PROOF. Elementary. □

LEMMA 1.1.8. *Let S and T be non-empty subsets of a complete lattice L, and let*

$$X = \{\, s \wedge t \mid s \in S,\, t \in T \,\}, \quad Y = \{\, s \vee t \mid s \in S,\, t \in T \,\}.$$

Then $\bigwedge X = (\bigwedge S) \wedge (\bigwedge T)$ and $\bigvee Y = (\bigvee S) \vee (\bigvee T)$.

PROOF. For all $s \in S$ and $t \in T$, $s \vee t \leqslant (\bigvee S) \vee (\bigvee T)$, and hence

$$\bigvee Y \leqslant \big(\bigvee S\big) \vee \big(\bigvee T\big).$$

On the other hand, for all $s \in S$ and $t \in T$, $s \leqslant s \vee t \leqslant \bigvee Y$ and $t \leqslant s \vee t \leqslant \bigvee Y$, so that $\bigvee S \leqslant \bigvee Y$ and $\bigvee T \leqslant \bigvee Y$, giving $(\bigvee S) \vee (\bigvee T) \leqslant \bigvee Y$. Thus $\bigvee Y = (\bigvee S) \vee (\bigvee T)$. Similarly, $\bigwedge X = (\bigwedge S) \wedge (\bigwedge T)$. □

Note that Lemmas 1.1.7 and 1.1.8 are true for any finite subsets of an arbitrary lattice L.

Upper continuous lattices

A lattice L is said to be *upper continuous* if L is complete and

$$a \wedge \big(\bigvee S\big) = \bigvee\{\, a \wedge s \mid s \in S \,\},$$

for every $a \in L$ and every chain (or, equivalently, upward directed subset) $S \subseteq L$. Recall that a subset S of L is said to be *upward directed* if for every $x, y \in S$ there exists $z \in S$ with $x \leqslant z$ and $y \leqslant z$. There is an analogous definition for a *downward directed* set. A lattice L is called *lower continuous* if L is complete and

$$a \vee \big(\bigwedge S\big) = \bigwedge\{\, a \vee s \mid s \in S \,\},$$

for every $a \in L$ and every chain (or, equivalently, downward directed subset) $S \subseteq L$. Note that a lattice L is lower continuous if and only if its opposite lattice L^0 is upper continuous.

The lattice $\mathcal{L}(M)$ of all submodules of a right R-module M is upper continuous. However, $\mathcal{L}(M)$ need not be lower continuous. For example, in the case $R = \mathbb{Z}$, the ring of rational integers, let $A = \mathbb{Z}2$ and let $\mathcal{C}$ be the chain of ideals $\mathbb{Z}3 \supseteq \mathbb{Z}9 \supseteq \mathbb{Z}27 \supseteq \dots$. Then $\bigwedge \mathcal{C} = 0$ and $A \vee (\bigwedge \mathcal{C}) = A$. However, $A \vee C = \mathbb{Z}$ for all $C \in \mathcal{C}$, so that if $\mathcal{D} = \{\, A \vee C \mid C \in \mathcal{C}\}$, then $\bigwedge \mathcal{D} = \mathbb{Z}$. Thus $\mathcal{L}(\mathbb{Z})$ is not lower continuous.

Given any ring R, an R-module M is called *linearly compact* provided for every non-empty family $(m_i)_{i \in I}$ of elements of M and family of submodules $(N_i)_{i \in I}$ of M with $\bigcap_{i \in J}(m_i + N_i) \neq \varnothing$ for every finite subset J of I, then $\bigcap_{i \in I}(m_i + N_i) \neq \varnothing$. Every linearly compact R-module M has the property that its associated submodule lattice $\mathcal{L}(M)$ is lower continuous.

1.2. Essential and other special elements

In this section we present some basic concepts of Lattice Theory like essential element, closed element, pseudo-complement element, E-complemented lattice, pseudo-complemented lattice, strongly pseudo-complemented lattice, and essentially complemented lattice, that will be frequently used in this text.

Throughout this section L will denote a modular lattice with least element 0, i.e., $L \in \mathcal{M}_0$.

Essential elements

An element $e \in L$ is called *essential* (*in* L) if $e \wedge a \neq 0$ for all $0 \neq a \in L$, and $E(L)$ will denote the set of all essential elements of L. Clearly, if L has a greatest element 1 then 1 is an essential element of L. However, in general L need not possess essential elements; e.g., in Example 1.1.6, the only essential element of H is V, and $E(G) = \varnothing$.

We list below some elementary properties of essential elements of a lattice L (in case $E(L)$ is non-empty).

LEMMA 1.2.1. *Let $e \leqslant b$ in L. Then $e \in E(L) \iff e \in E(b/0)$ and $b \in E(L)$.*

PROOF. Elementary. □

LEMMA 1.2.2. *Let $e_i \in E(L)\,(1 \leqslant i \leqslant n)$, for some $n \in \mathbb{N}$. Then*
(1) *$e_1 \wedge \cdots \wedge e_n \in E(L)$.*
(2) *$a \wedge e_1 \wedge \cdots \wedge e_n \in E(a/0)$ for every $a \in L$.*

PROOF. Elementary. □

COROLLARY 1.2.3. *$E(L)$ is a sublattice of L.*

PROOF. By Lemmas 1.2.1 and 1.2.2. □

LEMMA 1.2.4. *Let $n \in \mathbb{N}$, let $a_i \in L$, and let $e_i \in E(a_i/0)$ for all $1 \leqslant i \leqslant n$. Then*

$$e_1 \wedge \ldots \wedge e_n \in E((a_1 \wedge \cdots \wedge a_n)/0).$$

PROOF. Elementary. □

COROLLARY 1.2.5. *Let $a, b, c \in L$ be such that $a \in E(b/0)$. Then*

$$a \wedge c \in E((b \wedge c)/0).$$

PROOF. By Lemma 1.2.4. □

In contrast to Lemma 1.2.4, in general, if $n \in \mathbb{N}$, $a_i \in L$ and $e_i \in E(a_i/0)$ for all $1 \leqslant i \leqslant n$, then it does not follow that $e_1 \vee \cdots \vee e_n \in E((a_1 \vee \cdots \vee a_n)/0)$. However, as we shall show next, this is the case in some circumstances. First we prove the following two lemmas.

LEMMA 1.2.6. *Let $a, b, c \in L$ be such that $a \wedge b = 0$ and $(a \vee b) \wedge c = 0$. Then $(a \vee c) \wedge b = 0$.*

PROOF. By modularity, we have

$$(a\vee c)\wedge b = (a\vee c)\wedge((a\vee b)\wedge b) = ((a\vee c)\wedge(a\vee b))\wedge b = (a\vee((a\vee b)\wedge c))\wedge b = a\wedge b = 0. \quad \square$$

Lemma 1.2.7. *Let $a, b, c \in L$ be such that $a \wedge b = 0$ and $c \in E(b/0)$. Then $a \vee c \in E((a \vee b)/0)$.*

Proof. Let $x \in (a \vee b)/0$ be such that $(a \vee c) \wedge x = 0$. Because $a \wedge c = 0$, Lemma 1.2.6 gives that $(a \vee x) \wedge c = 0$ and hence $(a \vee x) \wedge b = 0$. But $a \wedge x = 0$, so that, again using Lemma 1.2.6, $(a \vee b) \wedge x = 0$. It follows that $x = 0$. Hence $a \vee c \in E((a \vee b)/0)$. $\square$

Corollary 1.2.8. *Let a_i, $b_i \in L$ $(i = 1, 2)$ be such that $a_i \in E(b_i/0)$ $(i = 1, 2)$. Then $b_1 \wedge b_2 = 0 \iff a_1 \wedge a_2 = 0$. Moreover, if $a_1 \wedge a_2 = 0$ then $a_1 \vee a_2 \in E((b_1 \vee b_2)/0)$.*

Proof. The first part is elementary and the second is a consequence of Lemmas 1.2.1 and 1.2.7. $\square$

This brings us to another key concept. A non-empty subset S of L is called *independent* if $0 \notin S$, and for every $x \in S$, positive integer n, and subset $T = \{t_1, \ldots, t_n\}$ of S with $x \notin T$, we have

$$x \wedge (t_1 \vee \cdots \vee t_n) = 0.$$

Clearly a subset S of L is independent if and only if every finite subset of S is independent.

Notice that a slightly different definition for independence in lattices is given in [**44**] as follows: a subset S of a complete lattice L is said to be *join independent*, or just *independent*, if $0 \notin S$ and $s \wedge \bigvee(S \setminus \{s\}) = 0$ for all $s \in S$. In case L is upper continuous, then $S \subseteq L$ is independent if and only if every finite subset of S is independent, so this definition agrees with ours. Alternatively, we say that a family $(x_i)_{i \in I}$ of elements of a complete lattice L is *independent* if $x_i \neq 0$ and $x_i \wedge (\bigvee_{j \in I \setminus \{i\}} x_j) = 0$ for every $i \in I$, and in this case, necessarily $x_p \neq x_q$ for each $p \neq q$ in I. Thus, the definitions of independence, using subsets or families of elements of L, are essentially the same.

For a non-empty subset S of L, we shall use throughout this text the *direct join* notation $a = \dot{\bigvee}_{b \in S} b$ (or $a = \dot{\bigvee} S$) if $a = \bigvee S$ and S is an independent subset of L.

Proposition 1.2.9. *Let $n \in \mathbb{N}$, let $\{b_1, \ldots, b_n\}$ be an independent subset of L, and let $a_i \in E(b_i/0)$ $(1 \leqslant i \leqslant n)$. Then $a_1 \vee \cdots \vee a_n \in E((b_1 \vee \cdots \vee b_n)/0)$.*

Proof. By Corollary 1.2.8 and induction on n. $\square$

Lemma 1.2.10. *$E([a)) \subseteq E(L)$ for every $a \in L$.*

Proof. Let $b \in E([a))$. Suppose that $c \in L$ and $b \wedge c = 0$. Then $b \wedge (a \vee c) = a \vee (b \wedge c) = a$, so that $a \vee c = a$. Hence $c \leqslant a \leqslant b$ and $c = b \wedge c = 0$. Thus $b \in E(L)$. $\square$

Closed elements

By a *closed element* of a lattice L we mean an element c such that whenever $a \in L$ with $c \leqslant a$ and $c \in E(a/0)$ then $a = c$. We denote by $C(L)$ the set of all closed elements of L. Note that 0 is a closed element of L.

In Example 1.1.6 every element of H and every element of G is closed. Note further that every element of a complemented lattice is closed. However, if R is any unital ring, U a simple right R-module, and E the injective hull of U, then in the lattice $\mathcal{L}(E)$ of all submodules of E the only closed elements are 0 and E. In case R is the ring $\mathbb{Z}$ of rational integers and W the sublattice of $\mathcal{L}(E)$ consisting of all finitely generated submodules of E, then 0 is the only closed element of W. Thus $C(H) = H$, $C(\mathcal{L}(E)) = \{0, E\}$ and $C(W) = \{0\}$.

Note that if K is a sublattice of L and c is an element of K such that $c \in C(L)$ then it is clear that $c \in C(K)$. In other words, $K \cap C(L) \subseteq C(K)$ for any sublattice K of L.

Lemma 1.2.11. *Let c be an element of a lattice L. If $c \in C(L)$ then*

$$[c) \cap E(L) \subseteq E([c)).$$

Proof. Let $a \in [c) \cap E(L)$. Let $b \in [c)$ with $b \neq c$. Then $c \notin E(b/0)$ because $c \in C(L)$. There exists $0 \neq d \leqslant b$ with $c \wedge d = 0$. Then $a \wedge d \neq 0$. Also,

$$a \wedge (c \vee d) = c \vee (a \wedge d) > c$$

for otherwise $a \wedge d \leqslant c \wedge d = 0$, a contradiction. Thus $a \wedge b > c$. It follows that $a \in E([c))$. □

E-Complemented lattices

We shall show that the converse of Lemma 1.2.11 is true for so called E-complemented lattices. A lattice L is called *E-complemented* (E for "Essential") provided for each $a \in L$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b \in E(L)$. Clearly, any complemented lattice is E-complemented. In fact, a lattice L is complemented if and only if L is E-complemented and $E(L) = \{1\}$.

Before proving the converse of Lemma 1.2.11 for E-complemented lattices, we first obtain some information about such lattices.

Lemma 1.2.12. *The following statements hold for an E-complemented lattice L.*

(1) *$[a)$ is E-complemented for every $a \in C(L)$.*
(2) *b/a is E-complemented for all elements $a \leqslant b$ in L with $a \in C(L)$.*
(3) *For any $b, c \in L$ with $b \wedge c = 0$ there exists $d \in L$ such that $c \leqslant d$, $b \wedge d = 0$, and $b \vee d \in E(L)$.*

Proof. The proof of (1) is similar to the proof of (2) and so we shall prove (2). Let $x \in b/a$. By hypothesis there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in E(L)$. Let $z := a \vee (b \wedge y) \in b/a$. Note that $x \wedge z = a$. Now we show that $x \vee z \in E(b/a)$. Note that $x \vee y \in [a) \cap E(L) \subseteq E([a))$, by Lemma 1.2.11. Now, $x \vee z = b \wedge (x \vee y) \in E(b/a)$ by Lemma 1.2.2. It follows that b/a is E-complemented.

(3) There exists $f \in L$ such that $(b \vee c) \wedge f = 0$ and $b \vee c \vee f \in E(L)$. By Lemma 1.2.6, $b \wedge (c \vee f) = 0$ and hence $d := c \vee f$ has the desired properties. □

Proposition 1.2.13. *Let c be an element of an E-complemented lattice L. Then*

$$c \in C(L) \iff [c) \cap E(L) \subseteq E([c)).$$

Proof. The necessity is proved in Lemma 1.2.11. Conversely, suppose that

$$[c) \cap E(L) \subseteq E([c)).$$

Now, let $x \in L$ with $c \leqslant x$ and $c \in E(x/0)$. Because L is E-complemented, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in E(L)$. By Lemmas 1.2.1 and 1.2.7, $c \vee y \in E(L)$, and so, by hypothesis, $c \vee y \in E([c))$. Now $(c \vee y) \wedge x = c \vee (y \wedge x) = c$, so that $x = c$. It follows that $c \in C(L)$. □

Corollary 1.2.14. *Let L be any E-complemented lattice and let $b, c \in L$ be such that $c \leqslant b$, $c \in C(b/0)$, and $b \in C(L)$. Then $c \in C(L)$.*

Proof. Because L is E-complemented, there exists $b' \in L$ such that $b \wedge b' = 0$ and $b \vee b' \in E(L)$. Also, by Lemma 1.2.12(2), there exists $c' \in b/0$ such that $c \wedge c' = 0$ and $c \vee c' \in E(b/0)$. Next, by Proposition 1.2.13, $b \vee b' \in E([b))$ and hence, by Lemma 1.2.10, $b \vee b' \in E([c))$. Now $c \vee c' \in E(b/c)$ by Proposition 1.2.13. But $b \wedge (b' \vee c) = c \vee (b \wedge b') = c$. Applying Lemma 1.2.7, $c \vee c' \vee b' \in E((b \vee b')/c))$. Next, Lemma 1.2.1 gives that $c \vee c' \vee b' \in E([c))$.

Let $c \leqslant d$ in L with $c \in E(d/0)$. We want to prove that $c = d$. Since $(c \vee c') \wedge b' = 0$ it follows that $c \wedge (c' \vee b') = 0$ (Lemma 1.2.6). Thus $d \wedge (c' \vee b') = 0$ and this implies that $d \wedge (c \vee c' \vee b') = c \vee (d \wedge (c' \vee b')) = c$. Since $c \vee c' \vee b' \in E([c))$ it follows that $c = d$, as required. Thus $c \in C(L)$. □

Pseudo-complements

Next we introduce a special subset of $C(L)$ and a special class of E-complemented lattices. Given an element $a \in L$, an element $b \in L$ is called a *pseudo-complement* of a (in L) provided b is maximal in the set of all elements c in L such that $a \wedge c = 0$. By a *pseudo-complement* of L we mean any element $b \in L$ such that b is a pseudo-complement (in L) of some element $a \in L$. We shall denote by $P(L)$ the set of all pseudo-complements of L. The lattice L is called *pseudo-complemented* if every element a has a pseudo-complement.

Clearly if L has a greatest element, then every complement of $a \in L$ is a pseudo-complement of a and any complemented lattice is pseudo-complemented.

Lemma 1.2.15. *Let b be a pseudo-complement of $a \in L$. Then $a \vee b \in E(L)$.*

Proof. Suppose that $(a \vee b) \wedge c = 0$, for some $c \in L$. Then Lemma 1.2.6 gives that $(b \vee c) \wedge a = 0$, and hence $b = b \vee c$, i.e., $c \leqslant b$. But this implies that

$$c = c \wedge b \leqslant c \wedge (a \vee b) = 0.$$

It follows that $a \vee b \in E(L)$. □

Proposition 1.2.16. *Let c be an element of a lattice L. Then $c \in P(L)$ if and only if $c \in C(L)$ and there exists an element $a \in L$ such that $a \wedge c = 0$ and $a \vee c \in E(L)$. So, $P(L) \subseteq C(L)$.*

Proof. Let $c \in P(L)$. There exists an element $a \in L$ such that c is a pseudo-complement of a. Suppose that $c \in E(b/0)$ for some $b \in L$. Then $a \wedge b = 0$ and hence $c = b$. Thus $c \in C(L)$. Moreover, $a \wedge c = 0$ and, by Lemma 1.2.15, $a \vee c \in E(L)$.

Conversely, suppose that c has the stated properties. We claim that c is a pseudo-complement of a in L. Suppose not. Then there exists $d \in L$ such that $c < d$ and $a \wedge d = 0$. Now $c \notin E(d/0)$ so that there exists $0 \neq x \leqslant d$ with $c \wedge x = 0$. Next, $(c \vee x) \wedge a = 0$ gives that $(a \vee c) \wedge x = 0$ by Lemma 1.2.6, and hence $x = 0$, which is a contradiction. Thus c is a pseudo-complement of a in L and, in particular, $c \in P(L)$. □

Note that the inclusion $P(L) \subseteq C(L)$ in Proposition 1.2.16 may be strict: indeed, in Example 1.1.6, $C(G) = G$ but $P(G) = \varnothing$. However, this is not the case for E-complemented lattices, as the next result shows.

Corollary 1.2.17. *If L is any E-complemented lattice then $P(L) = C(L)$.*

Proof. By Proposition 1.2.16. □

Proposition 1.2.18. *The following assertions hold.*

(1) *Every pseudo-complemented lattice is E-complemented.*
(2) *Every E-complemented lattice L such that $E(L)$ is Noetherian is pseudo-complemented.*

Proof. (1) By Lemma 1.2.15.

(2) Let $a \in L$. By hypothesis, there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b \in E(L)$. Suppose that there exists $b_1 \in L$ with $b < b_1$ and $a \wedge b_1 = 0$. Now suppose that $a \vee b = a \vee b_1$. Then

$$b_1 = b_1 \wedge (a \vee b_1) = b_1 \wedge (a \vee b) = b \vee (a \wedge b_1) = b \vee 0 = b,$$

which is a contradiction. Thus $a \vee b < a \vee b_1$. If now $b_1 < b_2$ and $a \wedge b_2 = 0$ for some $b_2 \in L$, then the above argument gives $a \vee b_1 < a \vee b_2$. Repeat this argument. This gives an ascending chain $a \vee b < a \vee b_1 < a \vee b_2 < \dots$ in $E(L)$. Because $E(L)$ is Noetherian, this process must stop in a finite number of steps. Thus, there exists $n \in \mathbb{N}$ such that b_n is a pseudo-complement of a. It follows that L is pseudo-complemented. □

Strongly pseudo-complemented lattices

A lattice L will be called *strongly pseudo-complemented* if, for all $a, b \in L$ with $a \wedge b = 0$, there exists a pseudo-complement p of a in L such that $b \leqslant p$. Clearly, strongly pseudo-complemented lattices are pseudo-complemented.

Let R be any unital ring, let M be a right R-module, and let A and B be submodules of M such that $A \cap B = 0$. By Zorn's Lemma, the set of all submodules Q of M such that $B \subseteq Q$ and $A \cap Q = 0$ has a maximal member. Thus, the lattice $\mathcal{L}(M)$ of all submodules of M is strongly pseudo-complemented, and so, it is pseudo-complemented and E-complemented (Proposition 1.2.18).

Lemma 1.2.19. *Any complemented lattice L is strongly pseudo-complemented.*

Proof. Let $a, b \in L$ with $a \wedge b = 0$. Because the sublattice $1/b$ is complemented by Proposition 1.1.5, there exists $c \in L$ such that $1 = (a \vee b) \vee c$ and $b = (a \vee b) \wedge c$. It follows that $1 = a \vee c$ and $a \wedge c = a \wedge (a \vee b) \wedge c = a \wedge b = 0$. Thus c is a pseudo-complement of a in L and $b \leqslant c$. □

In Example 1.1.6, $P(H) = H$ and H is pseudo-complemented, but $P(G) = \varnothing$ and G is not pseudo-complemented. Thus, in general, sublattices of pseudo-complemented lattices need not be pseudo-complemented. However, note the following fact.

Lemma 1.2.20. *Let a be any element of a (strongly) pseudo-complemented lattice L. Then the sublattice $a/0$ is also (strongly) pseudo-complemented.*

Proof. We shall prove the result for strongly pseudo-complemented lattices as the proof for pseudo-complemented lattices is similar. Suppose that L is strongly pseudo-complemented. Let $a \in L$ and let $b, c \in a/0$ with $b \wedge c = 0$. By hypothesis, there exists a pseudo-complement p of b in L with $c \leqslant p$. Note that $a \wedge p \in a/0$, $c \leqslant a \wedge p$, and $b \wedge (a \wedge p) = b \wedge p = 0$. Suppose that $a \wedge p \leqslant d \in a/0$ and $b \wedge d = 0$. Then

$$b \wedge (p \vee d) = b \wedge a \wedge (p \vee d) = b \wedge (d \vee (a \wedge p)) = b \wedge d = 0,$$

and hence $p = p \vee d$. It follows that $d \leqslant p$, and hence $d = a \wedge p$. Thus $a \wedge p$ is a pseudo-complement of b in $a/0$. It follows that the sublattice $a/0$ is strongly pseudo-complemented. □

We have seen above that if L is a pseudo-complemented lattice then so too is the sublattice $a/0$ for every $a \in L$. Moreover, every strongly pseudo-complemented lattice is pseudo-complemented. These observations bring us to the following result.

Proposition 1.2.21. *Let L be a lattice such that the sublattice $[a)$ is pseudo-complemented for every $a \in L$. Then L is strongly pseudo-complemented.*

Proof. Let $a, b \in L$ be such that $a \wedge b = 0$. Because $[b)$ is pseudo-complemented, there exists an element p of L maximal with respect to the property $(a \vee b) \wedge p = b$. Note that $a \wedge p \leqslant (a \vee b) \wedge p = b$, so that $a \wedge p \leqslant a \wedge b = 0$. Thus $a \wedge p = 0$. Suppose that $a \wedge q = 0$ for some $q \in L$ with $p < q$. Then $(a \vee b) \wedge q = b \vee (a \wedge q) = b$ and hence $p = q$. Thus p is a pseudo-complement of a such that $b \leqslant p$. It follows that L is strongly pseudo-complemented. □

A lattice L such that $[a)$ is E-complemented for every $a \in L$ is called *completely E-complemented.* These lattices will be amply discussed in Section 4.3. More generally, if $\mathbb{P}$ is a property of posets, we say that a poset X is *completely* $\mathbb{P}$ if $[x)$ has the property $\mathbb{P}$ for all $x \in X$.

Thus, Proposition 1.2.21 shows that if a lattice L is completely pseudo-complemented then L is completely strongly pseudo-complemented.

Essentially closed lattices

A lattice L will be called *essentially closed* if for each element c in L there exists an element $e \in L$ maximal in the set of elements $f \in L$ such that $c \in E(f/0)$.

Let R be a ring and let M be a right R-module. For any submodule A of M, let $\mathcal{S}_A$ denote the set of all submodules B of M such that A is an essential submodule of B (i.e., in the lattice $\mathcal{L}(M)$ of all submodules of M, $A \in E(B/0)$). Clearly, A belongs to $\mathcal{S}_A$ and, by Zorn's Lemma, $\mathcal{S}_A$ has a maximal member C. Thus $\mathcal{L}(M)$ is essentially closed. Note the following characterization of essentially closed lattices.

Proposition 1.2.22. *A lattice L is essentially closed if and only if for each $a \in L$ there exists $c \in C(L)$ such that $a \in E(c/0)$.*

Proof. Suppose first that L is essentially closed. Let $a \in L$. There exists an element c in L which is maximal with respect to the property $a \in E(c/0)$. Let $b \in L$ such that $c \in E(b/0)$. By Lemma 1.2.1, $a \in E(b/0)$, so that $c = b$ by the choice of c. Thus $c \in C(L)$.

Conversely, assume that L has the stated property, and let $x \in L$. Then there exists $y \in C(L)$ such that $x \in E(y/0)$. Suppose that $x \in E(z/0)$ for some $z \in L$ with $y \leqslant z$. Then $y \in E(z/0)$ by Lemma 1.2.1, so that $y = z$. It follows that L is essentially closed. □

Corollary 1.2.23. *Let L be an essentially closed lattice. Then $[c)$ is essentially closed for every $c \in C(L)$.*

Proof. Let $c \in C(L)$. Next let $x \in [c)$. By Proposition 1.2.22, there exists $y \in C(L)$ such that $x \in E(y/0)$. Thus $y \in [c) \cap C(L) \subseteq C([c))$ as we remarked before Lemma 1.2.11. But $[c) \cap E(y/0) \subseteq E(y/c)$ by Lemma 1.2.11, so that $x \in E(y/c)$. Again using Proposition 1.2.22 we see that $[c)$ is essentially closed. □

Theorem 1.2.24. *A lattice L is strongly pseudo-complemented if and only if L is E-complemented and essentially closed.*

Proof. Suppose first that L is strongly pseudo-complemented. By Proposition 1.2.18, L is E-complemented. Let $a \in L$. Since L is pseudo-complemented it follows that there exists an element $b \in L$ such that b is a pseudo-complement of a. Next, because L is strongly pseudo-complemented, there exists a pseudo-complement c of b such that $a \leqslant c$. By Proposition 1.2.16, $c \in C(L)$. Suppose that $d \in c/0$ and $a \wedge d = 0$. Then $(a \vee d) \wedge b = 0$ gives that $a \wedge (b \vee d) = 0$ by Lemma 1.2.6, and hence $b = b \vee d$ and $d \leqslant b$. It follows that $d = d \wedge b \leqslant c \wedge b = 0$. Thus $a \in E(c/0)$. By Proposition 1.2.22, L is essentially closed.

Conversely, suppose that L is E-complemented and essentially closed. Let $x, y \in L$ with $x \wedge y = 0$. There exists $z \in L$ such that $(x \vee y) \wedge z = 0$ and $x \vee y \vee z \in E(L)$. By Lemma 1.2.6, $x \wedge (y \vee z) = 0$. There exists $w \in L$ maximal with respect to the property $y \vee z \in E(w/0)$. Then $x \wedge w = 0$. Suppose that $w < v \in L$ and $x \wedge v = 0$. By hypothesis, $y \vee z \notin E(v/0)$, so that $(y \vee z) \wedge u = 0$ for some $0 \neq u \in v/0$. But this implies that $w \wedge u = 0$. Now $x \wedge (w \vee u) \leqslant x \wedge v = 0$ so that, by Lemma 1.2.6, $(x \vee w) \wedge u = 0$. But $x \vee w \in E(L)$, and we obtain the contradiction $u = 0$. We have proved that w is a pseudo-complement of x in L with $y \leqslant w$. It follows that L is strongly pseudo-complemented. □

To summarize, for any lattice L we have the following implications:

$$\boxed{\text{complemented} \Longrightarrow \text{strongly pseudo-complemented} \Longrightarrow \text{pseudo-complemented} \Longrightarrow \text{E-complemented}}$$

and

$$\boxed{\text{E-complemented \& essentially closed} \Longleftrightarrow \text{strongly pseudo-complemented} \Longrightarrow \text{essentially closed}}$$

(see Proposition 1.2.18, Lemma 1.2.19, and Theorem 1.2.24). There are no other implications in general. For example, the lattice G in Example 1.1.6 is essentially closed but not E-complemented. We shall give further examples at the end of the next section.

As we noticed before Lemma 1.2.19, the lattice of $\mathcal{L}(M)$ of all submodules of any module M is strongly pseudo-complemented. More generally, using Zorn's Lemma, we deduce that any upper continuous modular lattice is strongly pseudo-complemented.

1.3. Basic concepts in opposite lattices

In the previous section we saw that the lattice $\mathcal{L}(M)$ of all submodules of any right R-module M is strongly pseudo-complemented and hence also pseudo-complemented and E-complemented. In addition, $\mathcal{L}(M)$ is essentially closed. The opposite lattice $\mathcal{L}(M)^o$ is also modular by Proposition 1.1.3. Suppose that $\mathcal{L}(M)^o$ is E-complemented. This means that for any $A \leqslant M$ there exists $B \leqslant M$ such that $M = A + B$ and $A \cap B$ is a small submodule of M. Not every module has this property (for example, the $\mathbb{Z}$-module $\mathbb{Z}$ does not) but the modules which do have this property are called *weakly supplemented*.

Next, suppose that $\mathcal{L}(M)^o$ is pseudo-complemented. This means that for each submodule A of M there exists a submodule B of M which is minimal in the set of submodules C of M such that $M = A + C$. Modules M such that $\mathcal{L}(M)^o$ is pseudo-complemented are called *supplemented*.

Finally, a module M with the property that $\mathcal{L}(M)^o$ is strongly pseudo-complemented has the property that for all submodules A and B such that $M = A + B$ there exists a submodule C of B which is minimal with respect to the property $M = A + C$, and such a module M is called *amply supplemented*. Rings R with the property that every right R-module is (amply) supplemented are called *right perfect*. On the other hand, rings R with the property that every finitely generated right R-module is supplemented are called *semiperfect*.

Thus, applying our earlier results to $\mathcal{L}(M)^o$ we can obtain information about these different types of modules. This motivates what follows in this section. We do not intend to give the dual of every result in Section 1.2 but shall provide some information and leave the reader to deduce the rest.

Throughout this section L will denote a modular lattice with greatest element 1, i.e., $L \in \mathcal{M}_1$.

Small elements

Let L be a lattice with a greatest element 1. Note that L^o is a lattice with a least element and so is the type of lattice considered in the previous section. An element s of L is called *small* or *superfluous* (in L) provided $s \in E(L^o)$. Thus, a small element s of L is characterized by the fact that $1 \neq s \vee a$ for any element $a \in L$ with $a \neq 1$. We shall denote the set of small elements of L by $S(L)$, so that $S(L) = E(L^o)$ and $S(L)$ is a sublattice of L.

Lemma 1.3.1. *Let $a \leqslant s$ in L. Then $s \in S(L) \iff s \in S(1/a)$ and $a \in S(L)$.*

Proof. Apply Lemma 1.2.1 to the lattice L^o. □

Lemma 1.3.2. *Let $s_i \in S(L)\,(1 \leqslant i \leqslant n)$, for some positive integer n, Then*
(1) *$s_1 \vee \cdots \vee s_n \in S(L)$.*
(2) *$a \vee s_1 \vee \cdots \vee s_n \in S(1/a)$ for every $a \in L$.*

Proof. Apply Lemma 1.2.2 to the lattice L^o. □

By a *coindependent* subset X of L we mean an independent subset of L^o. Thus X is characterized by the property that $1 \notin X$ and for each $x \in X$, positive integer n, and subset $T=\{t_1,\dots,t_n\}$ of X with $x\notin T$, we have

$$x\vee(t_1\wedge\cdots\wedge t_n)=1.$$

PROPOSITION 1.3.3. *Let $n\in\mathbb{N}$, let $\{a_1,\dots,a_n\}$ be a coindependent subset of L, and let $s_i\in S(1/a_i)\,(1\leqslant i\leqslant n)$. Then*

$$s_1\wedge\cdots\wedge s_n\in S(1/(a_1\wedge\ldots\wedge a_n)).$$

PROOF. Apply Proposition 1.2.9 to the lattice L^o. □

Supplemented lattices

By a *coclosed* element of L we shall mean a closed element of L^o. Thus, c is a coclosed element of L if and only if for any $a\in L$ with $a\leqslant c$ and $c\in S(1/a)$ we have $c=a$. The lattice L will be called *weakly supplemented* provided for each element $a\in L$ there exists $b\in L$ such that $1=a\vee b$ and $a\wedge b\in S(L)$. Thus, L is weakly supplemented if and only if L^o is E-complemented.

PROPOSITION 1.3.4. *Let c be an element of a weakly supplemented lattice L. Then c is a coclosed element of L if and only if $(c]\cap S(L)\subseteq S((c])$.*

PROOF. Apply Proposition 1.2.13 to the lattice L^o. □

PROPOSITION 1.3.5. *Let $b\leqslant c$ in a weakly supplemented lattice L such that c is a coclosed element of $1/b$ and b is a coclosed element of L. Then c is a coclosed element of L.*

PROOF. Apply Corollary 1.2.14 to the lattice L^o. □

By a *supplement* (in L) of an element $a\in L$ we mean an element $b\in L$ such that b is minimal in the set of elements $c\in L$ with $1=a\vee c$. Thus, a is a supplement in L if and only if a is a pseudo-complement in L^o. If a has a complement, then, clearly, every complement of a is a supplement of a.

The lattice L is called *supplemented* provided every element of L has a supplement. Thus, the lattice L is supplemented if and only if L^o is pseudo-complemented.

PROPOSITION 1.3.6. *The following assertions hold for a lattice $L\in\mathcal{M}_1$.*

(1) *Every supplemented lattice is weakly supplemented.*
(2) *Every weakly supplemented lattice such that $S(L)$ is Artinian is supplemented.*

PROOF. Apply Proposition 1.2.18 to the lattice L^o. □

A lattice L will be called *amply supplemented* provided L^o is strongly pseudo-complemented, i.e., for all $a,b\in L$ with $1=a\vee b$, there exists a supplement c of a in L such that $c\leqslant b$.

We shall call a general lattice L *superfluously closed* if L^o is essentially closed, i.e., for each element $a\in L$ there exists an element $b\in L$ minimal with respect to the property that $a\in S(1/b)$. Clearly Artinian lattices are superfluously closed.

PROPOSITION 1.3.7. *A lattice L is superfluously closed if and only if for each $a\in L$ there exists a coclosed element s in L such that $a\in S(1/s)$.*

PROOF. Apply Proposition 1.2.22 to the lattice L^o. □

Amply supplemented lattices are obviously supplemented and can be characterized as follows.

THEOREM 1.3.8. *A lattice L is amply supplemented if and only if L is weakly supplemented and superfluously closed.*

PROOF. Apply Theorem 1.2.24 to the lattice L^o. $\square$

We began this section thinking of a right module M over a ring R. Theorem 1.3.8 can be restated for the lattice $\mathcal{L}(M)$ of all submodules of a module M as follows.

THEOREM 1.3.9. *Let R be any ring. Then a right R-module M is amply supplemented if and only if M is weakly supplemented and for each submodule A of M there exists a submodule B of M which is minimal with respect to the property that $(A+B)/B$ is a small submodule of M/B.* $\square$

We end this section with the following example.

EXAMPLE 1.3.10. By [**43**, Example 20.12] the $\mathbb{Z}$-module $\mathbb{Q}$ is weakly supplemented but not supplemented. Let L denote the lattice of all $\mathbb{Z}$-submodules of $\mathbb{Q}$. Then L is weakly supplemented but not supplemented, and so, the opposite lattice L^o is E-complemented but not pseudo-complemented. $\square$

CHAPTER 2

CHAIN CONDITIONS IN MODULAR LATTICES

Our main aim throughout this text is to investigate chains of elements in posets and, in particular, in rings, modules, categories, and lattices. In this chapter we study chain conditions in modular lattices. Specifically, we discuss Noetherian lattices, Artinian lattices, lattices with finite length, Goldie dimension of lattices, as well as Krull dimension and Gabriel dimension of arbitrary posets.

2.1. Noetherian and Artinian lattices

In this section we present some basic facts about Noetherian and Artinian lattices. As stressed before, all lattices considered throughout this text are supposed to be **modular**.

Basic properties

Recall from Chapter 1 that a poset P is Noetherian (respectively, Artinian) if for every ascending chain (respectively, descending chain)

$$x_1 \leqslant x_2 \leqslant x_3 \leqslant \dots \quad (\text{respectively, } \ x_1 \geqslant x_2 \geqslant x_3 \geqslant \dots\ )$$

of elements in P there exists an $n \in \mathbb{N}$ such that $x_n = x_{n+1} = \dots$.

Clearly, P is Noetherian if and only if the opposite poset P^o is Artinian.

Proposition 2.1.1. *Let a be an element of a lattice L. Then L is Noetherian (respectively, Artinian) if and only if $(a]$ and $[a)$ are both Noetherian (respectively, Artinian).*

Proof. By considering L^o, it is clearly sufficient to prove only the result in the Noetherian case. The necessity is clear.

Conversely, suppose that $(a]$ and $[a)$ are both Noetherian. Let $b_1 \leqslant b_2 \leqslant \dots$ be any ascending chain in L. Then $a \wedge b_1 \leqslant a \wedge b_2 \leqslant \dots$ is an ascending chain in $(a]$ and $a \vee b_1 \leqslant a \vee b_2 \leqslant \dots$ is an ascending chain in $[a)$. By hypothesis, there exists $m \in \mathbb{N}$ such that

$$a \wedge b_i = a \wedge b_{i+1} = \dots \qquad \text{and} \qquad a \vee b_i = a \vee b_{i+1} = \dots ,$$

for all $i \geqslant m$. Now let $i \geqslant m$. Then, by modularity, we have

$$b_{i+1} = b_{i+1} \wedge (a \vee b_{i+1}) = b_{i+1} \wedge (a \vee b_i) = b_i \vee (a \wedge b_{i+1}) = b_i \vee (a \wedge b_i) = b_i.$$

Thus $b_m = b_{m+1} = \dots$. It follows that L is Noetherian. $\square$

Corollary 2.1.2. *Let $n \in \mathbb{N}$, and let $a_1 \leqslant \dots \leqslant a_n$ be a finite ascending chain of elements of a lattice L. Then L is Noetherian (respectively, Artinian) if and only if $(a_1]$, $a_{i+1}/a_i\,(1 \leqslant i \leqslant n-1)$, and $[a_n)$ are all Noetherian (respectively, Artinian).*

Proof. Apply Proposition 2.1.1 and induction on n. $\square$

Corollary 2.1.3. *Let* $(a_i)_{1\leqslant i\leqslant n}$ *be a finite family of elements of a lattice* L. *Then*

(1) $[a_1 \wedge \ldots \wedge a_n)$ *is Noetherian* (*respectively, Artinian*) *if and only if* $[a_i)$ *is Noetherian* (*respectively, Artinian*) *for all* $1 \leqslant i \leqslant n$.

(2) $(a_1 \vee \cdots \vee a_n]$ *is Noetherian* (*respectively, Artinian*) *if and only if* $(a_i]$ *is Noetherian* (*respectively, Artinian*) *for all* $1 \leqslant i \leqslant n$.

Proof. (1) By induction on n, it is sufficient to prove the result only in the case $n = 2$. The necessity follows by Proposition 2.1.1. Conversely, suppose that $[a_1)$ and $[a_2)$ are both Noetherian. By Proposition 2.1.1, $(a_1 \vee a_2)/a_2$ is Noetherian. But, by Proposition 1.1.4, $(a_1 \vee a_2)/a_2 \simeq a_1/(a_1 \wedge a_2)$. Thus $[a_1)$ and $a_1/(a_1 \wedge a_2)$ are both Noetherian and, by Corollary 2.1.2, so too is $[a_1 \wedge a_2)$. Similarly, if $[a_1)$ and $[a_2)$ are both Artinian then so too is $[a_1 \wedge a_2)$.

(2) Apply (1) to the opposite lattice L^o. □

Proposition 2.1.4. *Every Noetherian* (*respectively, Artinian*) *lattice has a greatest* (*respectively, least*) *element.*

Proof. Let L be a Noetherian lattice. By Proposition 1.1.1, L has a maximal element, say a. Let $b \in L$. Then $a \leqslant a \vee b \in L$ so that $a = a \vee b$ and hence $b \leqslant a$. Thus a is the greatest element of L. For the Artinian result use L^o. □

Lattices of finite length

Let $a \leqslant b$ be elements of a lattice L. We say that the sublattice b/a of L is *simple* in case $a \neq b$ and $b/a = \{a, b\}$, i.e., the interval b/a has exactly two elements. If $a < b$ are elements of L and there is no $c \in L$ such that $a < c < b$, then we say that a is *covered* by b, and we write $a \prec b$. Thus, the interval b/a is *simple* if and only if $a \prec b$.

An element a of a lattice L with a least element 0 is said to be an *atom* of L if the interval $a/0$ is simple, or equivalently, if $0 \prec a$. As in [74], a lattice L with a greatest element 1 is called *semi-atomic* (respectively, *semi-Artinian*) if 1 is a join of atoms of L (respectively, if for every $x \in L$, $x \neq 1$, the sublattice $1/x$ of L contains an atom). The *socle* $\mathrm{Soc}(L)$ of a complete lattice L is the join of all atoms of L. If L has no atoms, then $\mathrm{Soc}\,(L) = 0$.

Notice that if M is a right R-module, then a submodule N of M is an atom in the lattice $\mathcal{L}(M)$ of all submodules of M if and only if N is a simple submodule of M. Moreover, the lattice $\mathcal{L}(M)$ is semi-atomic if and only if M is a semisimple module. As in the module case, if L is a semi-atomic upper continuous modular lattice, then L is complemented, and for every $a \leqslant b$ in L, the interval b/a of L is also a semi-atomic lattice by [74, Theorem 1.8.2 and Corollary 1.8.4].

More generally, by a *composition series* for b/a, if it exists, we mean a finite chain

$$a = c_0 < c_1 < \cdots < c_n = b,$$

for some $n \in \mathbb{N}$ and elements $c_i\,(0 \leqslant i \leqslant n)$ such that c_i/c_{i-1} is simple, i.e., $c_{i-1} \prec c_i$ for all $1 \leqslant i \leqslant n$. The integer n will be called the *length* of the series.

Given a non-zero lattice L with least element 0 and greatest element 1, we say that the lattice L *has a composition series* (or *has finite length*) in case $1/0$ has a composition series. Note that L has a composition series if and only if the opposite lattice L^o has a composition series. The next result characterizes when b/a has a composition series.

LEMMA 2.1.5. *Let $a < b$ be elements of a lattice L. Then b/a has a composition series if and only if b/a is both Noetherian and Artinian.*

PROOF. The necessity follows by Corollary 2.1.2. Conversely, suppose that b/a is both Noetherian and Artinian. Let b_1 be a maximal element in the set of proper elements of b/a. If $b_1 \neq a$ then choose b_2 maximal in the set of proper elements of b_1/a. Repeat this process to obtain a descending chain $b = b_0 > b_1 > b_2 \ldots \geqslant a$. Because b/a is Artinian, there exists $k \in \mathbb{N}$ such that $b_k = a$. Choose k as small as possible. Clearly, $a = b_k < b_{k-1} < \cdots < b_0 = b$ is a composition series for b/a. □

COROLLARY 2.1.6. *Let $a \leqslant c < d \leqslant b$ be elements of a lattice L such that b/a has a composition series. Then d/c has a composition series.*

PROOF. Apply Proposition 2.1.1 and Lemma 2.1.5. □

Given distinct elements $a < b$ of a lattice L such that b/a has a composition series, two composition series

$$a = c_0 < c_1 < \cdots < c_n = b,$$

and

$$a = d_0 < d_1 < \cdots < d_m = b$$

for b/a are called *equivalent* provided $m = n$ and there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $c_{\sigma(i)}/c_{\sigma(i)-1} \simeq d_i/d_{i-1}$ for all $1 \leqslant i \leqslant n$.

THEOREM 2.1.7. *Let $a < b$ be elements of a lattice L such that b/a has a composition series. Then any two composition series for b/a are equivalent, and in particular have the same length, called the* length *of b/a, and denoted by $\ell(b/a)$.*

PROOF. For any $c < d$ such that the interval d/c of L has a composition series we denote by $\lambda(d/c)$ the length of the shortest composition series for d/c.

Let $n = \lambda(b/a)$, and let

$$a = c_0 < c_1 < \cdots < c_n = b$$

be a composition series for b/a. We have to prove that any other composition series

$$a = d_0 < d_1 < \cdots < d_m = b$$

for b/a is equivalent to the former one.

We will proceed by induction on n. Suppose that $n = 1$. In this case, b/a is simple, hence $a < b$ is the unique composition series for b/a, and the result is proved. Now suppose that $n \geqslant 2$ and the result is true for any interval d/c of L with $\lambda(d/c) < n$.

If $c_{n-1} = d_{m-1}$ then

$$a = c_0 < c_1 < \cdots < c_{n-1} = d_{m-1},$$

and

$$a = d_0 < d_1 < \cdots < d_{m-1} = d_{m-1}$$

are two composition series for d_{m-1}/a with $\lambda(d_{m-1}/a) \leqslant n - 1$, and, by induction on n, they are equivalent, so that the two series for b/a considered above are equivalent.

Now suppose that $c_{n-1} \neq d_{m-1}$. It follows that $b = c_{n-1} \vee d_{m-1}$ because b/d_{m-1} is simple. By Corollary 2.1.6, the interval $(c_{n-1} \wedge d_{m-1})/a$ of L has a composition series, say

$$a = f_0 < f_1 < \cdots < f_k = c_{n-1} \wedge d_{m-1},$$

for some $k \in \mathbb{N}$. But $c_{n-1}/f_k \simeq d_m/d_{m-1}$ by Proposition 1.1.4, and hence

$$a = f_0 < f_1 < \cdots < f_k < c_{n-1}$$

is a composition series for c_{n-1}/a. By induction on n, the composition series

$$a = f_0 < f_1 < \cdots < f_k < c_{n-1}$$

and

$$a = c_0 < c_1 < \cdots < c_{n-1}$$

are equivalent. Similarly, the composition series

$$a = f_0 < f_1 < \cdots < f_k < d_{m-1}$$

and

$$a = d_0 < d_1 < \cdots < d_{m-1}$$

are equivalent.

Finally, note that Proposition 1.1.4 shows that

$$c_n/c_{n-1} \simeq d_{m-1}/f_k \quad \text{and} \quad d_m/d_{m-1} \simeq c_{n-1}/f_k.$$

Gathering this information together, we conclude that the given composition series for b/a are equivalent, as required. □

The next result is easily proved as in the module case:

PROPOSITION 2.1.8. *Let $L \in \mathcal{M}_{0,1}$, and let $a \in L$. Then L has finite length if and only if both intervals $a/0$ and $1/a$ have finite length, and in this case, we have*

$$\ell(L) = \ell(a/0) + \ell(1/a).$$ □

Compact elements

Throughout this subsection L will denote a complete modular lattice. An element $c \in L$ is called *compact* provided for any non-empty subset X of L with $c \leqslant \bigvee X$ there exists a finite non-empty subset Y of X such that $c \leqslant \bigvee Y$. One denotes by $K(L)$ the set of all compact elements of L.

The lattice L is said to be *compact* if its greatest element 1 is a compact element in L, and *compactly generated* if any element of L is a join of compact elements. Note that an element c of an upper continuous lattice is compact if and only if the lattice $c/0$ is compact (see Corollary 2.1.12), and any compactly generated lattice is upper continuous (see, e.g., [**85**, Chapter III, Proposition 5.3]).

We first aim to identify compact elements in lattices of submodules of modules. Let us mention that all rings considered in this text are associative with identity element, and all modules are unital right modules. We often write M_R to emphasize that M is a right module over the ring R. The notation $\mathcal{L}(M_R)$, or just $\mathcal{L}(M)$, stands for the lattice of all submodules of M, and the notation $N \leqslant M$ means that N is a submodule of M.

PROPOSITION 2.1.9. *The following statements hold for a right R-module M.*

(1) *A submodule C of M is a compact element of $\mathcal{L}(M)$ $\Longleftrightarrow$ C is a finitely generated submodule of M.*
(2) *$\mathcal{L}(M)$ is a compactly generated lattice.*

Proof. (1) Suppose first that C is finitely generated. Let X be a non-empty set of $\mathcal{L}(M)$ such that $C \leqslant \bigvee X$. This just means that the submodule C is contained in the submodule $\sum_{i\in I} A_i$ of M, where we indexed the set X of submodules of M as $X = \{\, A_i \,|\, i \in I \,\}$. Since C is finitely generated it follows that $C \subseteq \sum_{i\in J} A_i$ for some finite subset J of I. If Y is the finite subset $\{\, A_i \,|\, i \in J \,\}$ of X then, in $\mathcal{L}(M)$, we have $C \leqslant \bigvee Y$. It follows that C is a compact element of $\mathcal{L}(M)$.

Conversely, suppose that C is a compact element of $\mathcal{L}(M)$. Then, we can write C as the sum $C = \sum_{x\in C} xR$ of all its cyclic submodules xR, which means exactly that $C = \bigvee_{x\in C} xR$ in the lattice $\mathcal{L}(M)$. Since C is a compact element of $\mathcal{L}(M)$, it follows that $C \leqslant \bigvee_{x\in F} xR = \sum_{x\in F} xR \leqslant C$ for some finite subset F of C, and then $C = \sum_{x\in F} xR$, i.e., C is a finitely generated submodule of M, as required.

(2) follows from (1) because any submodule of M is the sum of all its cyclic submodules. $\square$

We can characterize compact elements in lattices in terms of chains.

Proposition 2.1.10. *An element c of L is compact if and only if for each chain X in L with $c \leqslant \bigvee X$ there exists $x \in X$ such that $c \leqslant x$.*

Proof. Suppose first that c is a compact element of L. Let X be any chain in L such that $c \leqslant \bigvee X$. By hypothesis, there exists a finite subset Y of X such that $c \leqslant \bigvee Y$. Because the finite set Y is a chain, there exists an element $u \in Y$ such that $y \leqslant u$ for all $y \in Y$. Then $c \leqslant \bigvee Y = u$ and $u \in X$.

Conversely, suppose that c has the stated condition relative to chains in L. Suppose that c is not a compact element of L. Then, there exists a non-empty subset Z of L such that $c \leqslant \bigvee Z$ but $c \nleqslant \bigvee W$ for every finite subset W of Z. Let S denote the set of all elements $a \in L$ such that $c \nleqslant a \vee (\bigvee W)$ for every finite subset W of Z. Note that $Z \subseteq S$ so that the set S is non-empty. Let C be a chain in S and let $b = \bigvee C$. Suppose that $b \notin S$. This implies that there exists a finite subset U of Z such that $c \leqslant b \vee (\bigvee U)$. Let $u = \bigvee U$. Then

$$c \leqslant b \vee u = \bigvee \{\, d \vee u \,|\, d \in C \,\},$$

by Lemma 1.1.8. But the set $\{\, d \vee u \,|\, d \in C\}$ is a chain of elements of L so that, by hypothesis, there exists $d_0 \in C$ such that

$$c \leqslant d_0 \vee u = d_0 \vee \Big(\bigvee U \Big),$$

a contradiction. Thus $b \in S$. So, we can apply Zorn's Lemma to deduce that S contains a maximal member m. If $z \leqslant m$ for all z in Z then $c \leqslant \bigvee Z \leqslant m$, a contradiction. Let $t \in Z$ such that $t \nleqslant m$. Then $m < m \vee t$ so that $m \vee t \notin S$. There exists a finite subset T of Z such that $c \leqslant (m \vee t) \vee (\bigvee T)$ so that $c \leqslant m \vee (\bigvee (T \cup \{t\}))$ which implies that $m \notin S$, a contradiction. Thus c is a compact element of L. $\square$

Corollary 2.1.11. *L is a compact lattice $\Longleftrightarrow$ $1 \neq \bigvee X$ for every chain X of proper elements of L.*

Proof. Apply Proposition 2.1.10. $\square$

We can extend Corollary 2.1.11 in the case of upper continuous lattices.

Corollary 2.1.12. *An element c of an upper continuous lattice L is compact $\Longleftrightarrow$ $c \neq \bigvee X$ for every chain X of proper elements of $c/0$ (i.e., $c/0$ is a compact lattice).*

Proof. The necessity follows by Proposition 2.1.10. Conversely, suppose that c has the stated condition. Let Y be a chain of elements of L such that $c \leqslant \bigvee Y$. Then

$$c = c \wedge \left(\bigvee Y\right) = \bigvee\{\, c \wedge y \mid y \in Y \,\},$$

because L is upper continuous. But $\{\, c \wedge y \mid y \in Y \,\}$ is a chain of elements of $c/0$. By hypothesis $c = c \wedge z \leqslant z$ for some $z \in Y$. By Proposition 2.1.10, it follows that c is a compact element. $\square$

One would expect compact elements of lattices to exhibit some of the properties of finitely generated submodules and this is indeed the case.

Lemma 2.1.13. *Let $n \in \mathbb{N}$, and let $c, c_1, \ldots, c_n \in K(L)$. Then, the following statements hold.*

(1) *$a \vee c \in K(1/a)$ for every $a \in L$.*
(2) *$c_1 \vee \cdots \vee c_n \in K(L)$.*
(3) *$K(1/c) \subseteq K(L)$, i.e., every compact element of $1/c$ is a compact element of L.*
(4) *For each $a \in L$ with $a < c$, there exists a maximal element m of $c/0 \setminus \{c\}$ such that $a \leqslant m$.*

Proof. (1) Let $a \in L$. Let X be any non-empty subset of $1/a$ with $a \vee c \leqslant \bigvee X$. Then $c \leqslant \bigvee X$, and hence $c \leqslant \bigvee Y$ for some finite subset Y of X. Clearly, $Y \subseteq 1/a$ implies that $a \vee c \leqslant \bigvee Y$. It follows that $a \vee c$ is a compact element of $1/a$.

(2) Let X be any non-empty subset of L such that $c_1 \vee \cdots \vee c_n \leqslant \bigvee X$. For each $1 \leqslant i \leqslant n$, $c_i \leqslant \bigvee X$ and hence there exists a finite subset Y_i of X such that $c_i \leqslant \bigvee Y_i$. If Y is the finite subset $Y_1 \cup \cdots \cup Y_n$ of X then $c_1 \vee \cdots \vee c_n \leqslant \bigvee Y$ by Lemma 1.1.7.

(3) Let b be any compact element of $1/c$, and let X be any subset of L such that $b \leqslant \bigvee X$. Set $X' := \{\, x \vee c \mid x \in X \,\}$. By hypothesis, there exists a finite subset U of X' such that $b \leqslant \bigvee U$. Next, there exists a finite subset Y of X such that every element u of U has the form $y \vee c$ for some $y \in Y$. But $c \leqslant b \leqslant \bigvee X$ so that $c \leqslant \bigvee Z$ for some finite subset Z of X. The set $Y \cup Z$ is a finite subset of X such that $b \leqslant \bigvee(Y \cup Z)$. It follows that b is a compact element of L.

(4) Let $a < c$, and set $S := \{\, b \in L \mid a \leqslant b < c \,\}$. Note that $a \in S$, so $S \neq \varnothing$. Let C be any chain contained in S. Suppose that $\bigvee C \notin S$. Then $c = \bigvee C$. By hypothesis, $c \leqslant \bigvee D$ for some finite subset D of C and hence $c \leqslant d$ for some $d \in D \subseteq C \subseteq S$. Then $c = d \in S$ a contradiction. Thus $\bigvee C \in S$, so we can apply Zorn's Lemma to obtain a maximal element of S, that is the desired element m. $\square$

Observe that assertion (4) in Lemma 2.1.13 is the latticial counterpart of the renowned *Krull Lemma* from Module Theory.

Proposition 2.1.14. *The following statements are equivalent for a lattice L.*

(1) *L is Noetherian.*
(2) *For each subset X of L there exists a finite subset Y of X with $\bigvee X = \bigvee Y$.*
(3) *Every element of L is compact.*

Proof. (1) $\Longrightarrow$(2) Let X be any (non-empty) subset of a Noetherian lattice L. Consider the non-empty set S of elements of L of the form $\bigvee U$, where U is a finite subset of X. By Proposition 1.1.1, there exists a finite subset Y of X such that $\bigvee Y$ is maximal in S. Let $x \in X$. If Z denotes the finite subset $Y \cup \{x\}$ of X then $\bigvee Y \leqslant \bigvee Z$, and hence $\bigvee Y = \bigvee Z$. It follows that $x \leqslant \bigvee Y$ for all $x \in X$, and so $\bigvee X \leqslant \bigvee Y$. Thus $\bigvee X = \bigvee Y$.

(2) $\Longrightarrow$ (3) Clear.

(3) $\Longrightarrow$ (1) Let $a_1 \leqslant a_2 \leqslant \ldots$ be any ascending chain of elements of L, and set

$$W := \{\, a_i \,|\, i \in \mathbb{N} \,\} \text{ and } b := \bigvee W.$$

By hypothesis, b is a compact element of L, and hence $b \leqslant \bigvee T$ for some finite subset T of W. But clearly $\bigvee T = a_n$ for some $n \in \mathbb{N}$. Thus, for each $i \geqslant n$,

$$a_i \leqslant \bigvee W = b \leqslant a_n,$$

so that $a_n = a_{n+1} = \ldots$. It follows that L is Noetherian. $\square$

Corollary 2.1.15. *Any Noetherian lattice L with least and greatest element is upper continuous.*

Proof. First, observe that by the proof of (1) $\Longrightarrow$ (2) in Proposition 2.1.14, any non-empty subset S of L has a least upper bound, and so, by [**85**, Chapter III, Proposition 1.2], L is a complete lattice.

Let C be a chain of L and $a \in L$. We have to prove that

$$a \wedge (\bigvee C) = \bigvee \{\, a \wedge c \,|\, c \in C \,\}.$$

By Proposition 2.1.14, we have $\bigvee C = \bigvee F$ for some finite subset F of C, and so $\bigvee C = b \in C$ because C is a chain. Then $a \wedge (\bigvee C) = a \wedge b = \bigvee \{a \wedge c) \,|\, c \in C\,\}$. $\square$

Compact lattices

In this subsection we present several results on compact lattices.

Proposition 2.1.16. *Let s be a small element of L. Then L is a compact lattice if and only if its interval $1/s$ is compact.*

Proof. The result can be reformulated as follows: 1 is compact in L if and only if 1 is compact in $1/s$. The necessity follows by Lemma 2.1.13(1). Conversely, suppose that 1 is a compact element of $1/s$. Let X be a chain of elements of L such that $1 = \bigvee X$, and set $Y := \{\, x \vee s \,|\, x \in X \,\}$. Then Y is a chain in $1/s$ such that $1 = \bigvee Y$. By Corollary 2.1.11, $1 = u \vee s$ for some $u \in X$ and hence $u = 1$. It follows that 1 is a compact element of L by Corollary 2.1.11 again. $\square$

Corollary 2.1.17. *Let s be a small element of L such that $1/s$ is Noetherian. Then L is a compact lattice.*

Proof. Apply Propositions 2.1.14 and 2.1.16. $\square$

Proposition 2.1.18. *The following statements hold for a lattice L.*

(1) *If L is a compact lattice, then so is also $d/0$ for any $d \in D(L)$. If additionally L is upper continuous, then $D(L) \subseteq K(L)$.*

(2) *If L is a compact lattice, then so is also any of its quotient intervals $1/a$.*
(3) *Assume that L is upper continuous, and let $a \leqslant b \leqslant c$ in L be such that the intervals b/a and c/b are both compact lattices. Then c/a is also a compact lattice.*
(4) *If L is a complete compact lattice, then any lattice isomorphic to L is also compact.*

Proof. (1) Let $d \in D(L)$. Then, there exists $d' \in L$ such that $d \vee d' = 1$ and $d \wedge d' = 0$. Let $A \subseteq L$ be such that $d = \bigvee_{x\in A} x$. Because 1 is a compact element of L and $1 = d \vee d' = \bigvee_{x\in A}(x \vee d')$, there exists a finite subset F of A such that

$$1 = \bigvee_{x\in F}(x \vee d') = (\bigvee_{x\in F} x) \vee d' = y \vee d',$$

where $y := \bigvee_{x\in F} x \leqslant d$. By modularity, we have

$$d = (y \vee d') \wedge d = y \vee (d \wedge d') = y \vee 0 = y = \bigvee_{x\in F} x,$$

which shows that d is a compact element of $d/0$.

Assume now that additionally L is upper continuous, and let $d \in D(L)$. We have just proved that d is a compact element of $d/0$. Then, by Corollary 2.1.12, d is a compact element of L. Thus $D(L) \subseteq K(L)$.

(2) is obvious.

(3) Without loss of generality we may assume that $a = 0$ and $c = 1$. So, we have to prove that if b is compact in $b/0$ and 1 is compact in $1/b$, then 1 is compact in $1/0 = L$. Let C be a chain in L such that $1 = \bigvee_{x\in C} x$. By upper continuity, we have $b = \bigvee_{x\in C}(x \wedge b)$, so $b = y \wedge b$, i.e., $b \leqslant y$ for some $y \in C$ because b is compact in $b/0$.

On the other hand, we have $1 = \bigvee_{x\in C}(x \vee b)$, so $1 = z \vee b$ for some $z \in C$ because 1 is compact in $1/b$. Because C is a chain, $t := y \vee z \in \{y, z\} \subseteq C$, so $b \leqslant y \leqslant t$ and $1 = t \vee b \leqslant t$, i.e., $1 = t \in C$, which shows that 1 is compact in L.

(4) Observe first that if $f : L \longrightarrow L'$ is a lattice isomorphism of complete lattices, then clearly

$$f(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} f(x_i)$$

for any family $(x_i)_{i\in I}$ of elements of L. This easily implies that $f(c) \in K(L')$ for any $c \in K(L)$; in particular $1' = f(1) \in K(L')$ if 1 is a compact element of L, that is, L' is compact if so is L. $\square$

Cocompact elements

Let L be a lattice with a greatest element 1. In this case, according to the usual definition in a poset, 1 is the unique maximal element of L. However, as in Module Theory, by a *maximal* element of L we mean an element which is maximal in the set of all proper elements of L. Thus, m is a maximal element of L if $m \neq 1$ and whenever $a \in L$ with $m \leqslant a$, then $a = m$ or $a = 1$. On the other hand, if L has a least element, then a *minimal* element of L is an element minimal in the set of all non-zero elements of L. Thus, m is a minimal element of L if $m \neq 0$ and whenever $a \in L$ with $a \leqslant m$ then $a = m$ or $a = 0$. An element c of a complete lattice L is called *cocompact* provided for every non-empty subset X of L with $\bigwedge X \leqslant c$ there exists a finite subset Y of X such that $\bigwedge Y \leqslant c$. Clearly c is a cocompact element of L if and only if c is a compact element of L^o. We wish to mention below just one result about cocompact elements. Notice that its proof involves the concept and properties of Goldie dimension which will be amply discussed in the next section.

THEOREM 2.1.19. *Let L be a non-zero upper continuous modular lattice. Then, the least element 0 of L is a cocompact element of L if and only if there exist $n \in \mathbb{N}$ and minimal elements $m_i\,(1 \leqslant i \leqslant n)$ of L such that $m_1 \vee \cdots \vee m_n \in E(L)$.*

PROOF. Suppose first that $m_1 \vee \cdots \vee m_n \in E(L)$ for some $n \in \mathbb{N}$ and minimal elements m_i $(1 \leqslant i \leqslant n)$ of L. If $a = m_1 \vee \cdots \vee m_n$, then $a/0$ is Artinian by Corollary 2.1.3. By Corollary 2.1.17 applied to the lattice L^o, it follows that its greatest element $0 \in L$ is a compact element of L^o, so 0 is a cocompact element of L, as desired.

Conversely, suppose that the element 0 is cocompact. We claim that L has finite Goldie dimension. If not, then L contains an infinite independent set $\{\, b_i \,|\, i \in \mathbb{N} \,\}$. Because 0 is a compact element of L^o, we can apply Lemma 2.1.13(4) for the lattice L^o to obtain for each $i \in \mathbb{N}$ a minimal element u_i of L such that $0 < u_i \leqslant b_i$. Note that the set $\{\, u_i \,|\, i \in \mathbb{N} \,\}$ is also independent. For each $j \in \mathbb{N}$, set

$$c_j := \bigvee_{i \geqslant j} u_i,$$

and consider the descending chain $c_1 \geqslant c_2 \geqslant \ldots$ of L. Let $C = \{\, c_i \,|\, i \in \mathbb{N} \,\}$, and set $c := \bigwedge C$. Since L is upper continuous by hypothesis, it is also strongly pseudo-complemented as we have observed just after Theorem 1.2.24, and, in particular, it is E-complemented by Theorem 1.2.24. So, there exists $x \in L$ such that $c \wedge x = 0$ and $e := c \vee x \in E(L)$. Since $(\bigwedge C) \wedge x = 0$ and 0 is cocompact, there exists a finite subset D of C such that $(\bigwedge D) \wedge x = 0$, and then $c_k \wedge x = 0$ for some $k \in \mathbb{N}$. For each $i \geqslant k$, $u_i \wedge e \neq 0$ because $e \in E(L)$, and hence $u_i = u_i \wedge e \leqslant e$ because u_i is a minimal element of L. It follows that $c_k \leqslant e = c \vee x$. Thus

$$c_k = c_k \wedge e = c_k \wedge (c \vee x) = c \vee (c_k \wedge x) = c \vee 0 = c \leqslant c_{k+1}.$$

This implies that $c_k = c_{k+1}$. But

$$u_{k+1} \leqslant u_{k+1} \vee u_{k+2} \leqslant \ldots \leqslant \bigvee_{j \geqslant k+1} u_j = c_{k+1},$$

so that, because L is upper continuous and $c_{k+1} = \bigvee_{i \geqslant k+1} u_i = \bigvee_{i \geqslant k+1} (u_{k+1} \vee \cdots \vee u_i)$ by a countable variant of Lemma 1.1.7, we have

$$u_k = u_k \wedge c_k = u_k \wedge c_{k+1} = \bigvee_{j \geqslant k+1} \big(u_k \wedge (u_{k+1} \vee \cdots \vee u_j)\big) = 0,$$

a contradiction. It follows that L has finite Goldie dimension, as claimed. By Corollary 2.2.5 there exists a finite independent set S of uniform elements $s_i\,(1 \leqslant i \leqslant n)$ of L such that $b := \bigvee S \in E(L)$, i.e., b is a small element of L^o. Since 0 is a cocompact element of L, the greatest element of L^o is a compact element of L^o, i.e., L^o is a compact lattice, so by Lemma 2.1.13(4) applied to the lattice L^o, for each $1 \leqslant i \leqslant n$ there exists a minimal element w_i of L such that $w_i \leqslant s_i$. Then Lemma 2.2.7 gives $w_1 \vee \cdots \vee w_n \in E(L)$, as required. □

If we apply Theorem 2.1.19 to the opposite lattice of a lattice L and translate the terms occurring in its statement in their dual counterparts, we obtain at once:

THEOREM 2.1.20. *A non-zero lower continuous modular lattice* L *is compact if and only if there exist* $n \in \mathbb{N}$ *and maximal elements* $m_i\,(1 \leqslant i \leqslant n)$ *of* L *such that* $m_1 \wedge \ldots \wedge m_n$ *is a small element of* L. □

Of course, all the results in this section have "duals" obtained by using the opposite lattice. No further proofs are required. In addition, results can be obtained for modules by applying the above results to the lattice of submodules of a module. Again we leave this to the reader.

2.2. Goldie dimension

A cornerstone in the development of modern Ring Theory is the concept of *Goldie dimension.* Modular lattices provide a very natural setting for the development of this dimension as we shall show in this section. Here our concern is with lattices which do not contain an infinite independent subset. We shall show that in this case, for any such lattice L, there exists a unique $n \in \mathbb{N} \cup \{0\}$, called the *Goldie dimension* of L, such that every independent subset of L contains at most n elements. Recall that we denote by $\mathbb{N}$ the set $\{1, 2, 3, \ldots\}$ of all natural numbers, and for any set S we denote the cardinality of S by $|S|$.

Throughout this section, excepting its last subsection, L will be a modular lattice with least element 0, i.e., $L \in \mathcal{M}_0$.

Some preparatory results

We first give a result which characterizes when a countable subset of L is independent.

LEMMA 2.2.1. *A subset* $S = \{\, a_i \mid i \in \mathbb{N} \,\}$ *of non-zero elements of* L *is independent if and only if*

$$a_{k+1} \wedge (a_1 \vee \cdots \vee a_k) = 0$$

for all $k \in \mathbb{N}$.

PROOF. The necessity is clear. Conversely, suppose that S has the stated property. For each $n \in \mathbb{N}$ set $S_n = \{a_1, \ldots, a_n\}$. We show that S_n is independent by induction on n. Clearly S_1 is independent. Suppose that S_n is independent for some $n \in \mathbb{N}$. Let $1 \leqslant i \leqslant n$ and set $b_i := a_1 \vee \cdots \vee a_{i-1} \vee a_{i+1} \vee \cdots \vee a_n$. Because S_n is independent we have $a_i \wedge b_i = 0$. But $a_{n+1} \wedge (a_i \vee b_i) = 0$ by hypothesis. Therefore, by Lemma 1.2.6, $a_i \wedge (b_i \vee a_{n+1}) = 0$. Since $a_{n+1} \wedge (a_1 \vee \cdots \vee a_n) = 0$, it follows that S_{n+1} is independent. Thus S_n is independent for every $n \in \mathbb{N}$ and therefore S is independent. Notice that for $n = 3$, the fact that S_3 is independent is exactly the result in Lemma 1.2.6. □

Given a finite subset S of L, recall that the greatest lower bound of S is denoted by $\bigwedge S$ and the least upper bound of S is denoted by $\bigvee S$.

COROLLARY 2.2.2. *Let* S *and* T *be non-empty finite subsets of* L. *Then* $S \cup T$ *is independent if and only if* S *and* T *are both independent and* $\left(\bigvee S\right) \wedge \left(\bigvee T\right) = 0$.

Proof. Suppose first that $S \cup T$ is independent. Clearly this implies that both S and T are independent. Assume that $T = \{t_1, \ldots, t_n\}$ for some $n \in \mathbb{N}$, and let $t := t_1 \vee \cdots \vee t_{n-1}$. We proceed by induction on n for any finite subset S of L. So, assume that $\left(\bigvee S\right) \wedge t = 0$. If we set $S' := S \cup \{t_1, \ldots, t_{n-1}\}$, then, because $S \cup T$ is independent by hypothesis, we have

$$0 = \left(\bigvee S'\right) \wedge t_n = \left(\left(\bigvee S\right) \vee t\right) \wedge t_n = 0.$$

By Lemma 1.2.6 it follows that $\left(\bigvee S\right) \wedge (t \vee t_n) = 0$. Thus $\left(\bigvee S\right) \wedge \left(\bigvee T\right) = 0$.

Conversely, suppose that S and T are independent and $\left(\bigvee S\right) \wedge \left(\bigvee T\right) = 0$. Then $\left(\bigvee S\right) \wedge t_1 = 0$ and $S \cup \{t_1\}$ is independent by Lemma 2.2.1. Now suppose that $S \cup \{t_1, \ldots, t_i\}$ is independent for some $1 \leqslant i \leqslant n-1$, and set $c := t_1 \vee \cdots \vee t_i$. Then $\left(\bigvee S\right) \wedge c = 0$ by the first part, and, by hypothesis, $\left(\bigvee S\right) \wedge (c \vee t_{i+1}) = 0$. Lemma 1.2.6 gives that $\left(\left(\bigvee S\right) \vee c\right) \wedge t_{i+1} = 0$. By Lemma 2.2.1 we deduce that $S \cup \{t_1, \ldots, t_{i+1}\}$ is independent. By induction, it follows that $S \cup T$ is independent. □

A lattice $L \in \mathcal{M}_0$ is said to be *uniform* if $L \neq 0$ and $x \wedge y \neq 0$ for any non-zero elements $x, y \in L$. An element u of a lattice L is called *uniform* if $u/0$ is a uniform lattice, i.e., if $u \neq 0$ and $a \wedge b \neq 0$ for all non-zero elements a and b in $u/0$. In other words, a non-zero element $u \in L$ is uniform if and only if every non-zero element of $u/0$ belongs to $E\,(u/0)$. We denote by $U(L)$ the set of all uniform elements of L.

Not every lattice contains uniform elements. For example, if R is a non-commutative domain which is not right Ore then the lattice $\mathcal{L}(R_R)$ of right ideals of R does not contain a uniform element. However, if L is a lattice which does not contain an infinite independent subset then not only does L contain a uniform element but it contains an abundance of uniform elements. This is what we aim to show in the next few results.

Lemma 2.2.3. *Let L be a lattice which does not contain an infinite independent set. Then $a/0$ contains a uniform element for every non-zero element a of L.*

Proof. Suppose not. Because a is not uniform, there exist $0 \neq a_1, b_1 \in a/0$ with $a_1 \wedge b_1 = 0$. Because b_1 is not uniform, there exist $0 \neq a_2, b_2 \in b_1/0$ such that $a_2 \wedge b_2 = 0$. Repeat this argument. Note that $a_n \neq 0\,(n \in \mathbb{N})$ and that

$$a_j \wedge (a_{j+1} \vee \cdots \vee a_i) \leqslant a_j \wedge b_j = 0,$$

for all $1 \leqslant j \leqslant i-1, i \geqslant 2$. By Lemma 2.2.1, $\{a_1, \ldots, a_i\}$ is independent for all $i \in \mathbb{N}$ and hence $\{a_1, a_2, a_3, \ldots\}$ is independent, a contradiction. The result follows. □

Lemma 2.2.4. *Let L be a non-zero lattice which does not contain an infinite independent subset, and let S be a (finite) independent set of uniform elements of L. Then there exists a subset T of L which is either the empty set or is a (finite) independent set of uniform elements of L such that $S \cup T$ is independent and $\bigvee\left(S \cup T\right) \in E(L)$.*

Proof. By Lemma 2.2.3, L contains a uniform element u_1. Suppose that S is the set $\{u_1, \ldots, u_n\}$ of uniform elements of L, for some $n \in \mathbb{N}$. Now suppose that $u_1 \vee \cdots \vee u_n \notin E(L)$. It follows that $(u_1 \vee \cdots \vee u_n) \wedge a = 0$ for some non-zero element a of L. Again using Lemma 2.2.3, there exists a uniform element u_{n+1} in $a/0$. Note that $(u_1 \vee \cdots \vee u_n) \wedge u_{n+1} = 0$. By Lemma 2.2.1, the set $\{u_1, \ldots, u_{n+1}\}$ is also independent. Repeat this argument. Since L does not contain an infinite independent subset it follows that there exist a positive integer $m \geqslant n$ and uniform elements $u_i\,(n+1 \leqslant i \leqslant m)$ such that the set $X = \{u_1, \dots, u_m\}$ is independent and $\bigvee X \in E(L)$. The result follows. $\square$

COROLLARY 2.2.5. *Let L be a non-zero lattice which does not contain an infinite independent subset. Then there exists a (finite) independent set S of uniform elements of L such that $\bigvee S \in E(L)$.*

PROOF. Apply Lemmas 2.2.3 and 2.2.4. $\square$

COROLLARY 2.2.6. *Let L be a non-zero lattice which does not contain an infinite independent subset. Then L is E-complemented.*

PROOF. Let $a \in L$. Suppose that $a \neq 0$. By Corollary 2.2.5, there exists a finite set of uniform elements of $a/0$ such that $\bigvee S \in E(a/0)$. Then Lemma 2.2.4 gives a set T such that $(\bigvee S) \wedge (\bigvee T) = 0$ and $(\bigvee S) \vee (\bigvee T) \in E(L)$. It follows that $a \wedge (\bigvee T) = 0$ and $a \vee (\bigvee T) \in E(L)$. In case $a = 0$ this proof can easily be adapted to find such a set T. It follows that L is E-complemented. $\square$

LEMMA 2.2.7. *Let L be a non-zero lattice which does not contain an infinite independent subset. Let S be any finite independent set of uniform elements of L such that $\bigvee S \in E(L)$, and let $e \in L$. Then $e \in E(L)$ if and only if $e \wedge s \neq 0$ for all $s \in S$.*

PROOF. The necessity is clear. Conversely, suppose that $S = \{u_1, \dots, u_n\}$ for some $n \in \mathbb{N}$, and that $e \wedge u_i \neq 0$ for all $1 \leqslant i \leqslant n$. For each $1 \leqslant i \leqslant n$, $e \wedge u_i \in E(u_i/0)$. By Proposition 1.2.9, $(e \wedge u_1) \vee \dots \vee (e \wedge u_n) \in E((u_1 \vee \dots \vee u_n)/0)$. Lemma 1.2.1 gives that $(e \wedge u_1) \vee \dots \vee (e \wedge u_n) \in E(L)$ and hence $e \in E(L)$. $\square$

The definition of the Goldie dimension

Recall that for any lattice $L \in \mathcal{M}_0$ we have denoted by $U(L)$ the set, possibly empty, of all uniform elements of L.

THEOREM 2.2.8. *Let $L \in \mathcal{M}_0$, and let S, T be two finite non-empty independent subsets of L. Then the following statements hold.*

(1) *If $S \subseteq U(L)$ and $\bigvee S \in E(L)$ then $|T| \leqslant |S|$.*
(2) *If $S \subseteq U(L)$, $T \subseteq U(L)$, $\bigvee S \in E(L)$, and $\bigvee T \in E(L)$ then $|S| = |T|$.*

PROOF. (1) Let $S = \{s_1, \dots, s_n\}$, $T = \{t_1, \dots, t_k\}$, and set

$$s := s_1 \vee \dots \vee s_n,\ t := t_1 \vee \dots \vee t_k.$$

Since $s \in E(L)$, we have $t'_i := t_i \wedge s \neq 0$, $t'_1 \vee \dots \vee t'_k \leqslant s$, and $\{t'_1, \dots, t'_k\}$ is an independent subset of $s/0$, so, without loss of generality, we may replace L by $s/0$, i.e., L has a greatest element $1 = s$. Set $\overline{t} := t_2 \vee \dots \vee t_k$. We claim that $\overline{t} \wedge s_j \neq 0$ for some $1 \leqslant j \leqslant n$. Assume not. Then, we would have $\overline{t} \wedge s_j \in E(s_j/0)$ for all $1 \leqslant j \leqslant n$ because $s_j \in U(L)$. By Proposition 1.2.9, we would obtain

$$(\overline{t} \wedge s_1) \vee \dots \vee (\overline{t} \wedge s_n) \in E((s_1 \vee \dots \vee s_n)/0) = E(s/0) = E(L).$$

But $(\overline{t} \wedge s_1) \vee \dots \vee (\overline{t} \wedge s_n) \leqslant \overline{t} \wedge s$, hence $\overline{t} \wedge s \in E(L)$, and so, by Lemma 1.2.1, $\overline{t} \in E(L)$, a contradiction. Therefore, by relabeling the s_i's, we may assume that $\overline{t} \wedge s_1 = 0$.

By Proposition 1.1.4, we have the following sequence of canonical lattice morphisms

$$\overline{t}/0 = \overline{t}/(\overline{t} \wedge s_1) \xrightarrow{\sim} (\overline{t} \vee s_1)/s_1 \hookrightarrow 1/s_1 \xrightarrow{\sim} (s_2 \vee \dots \vee s_n)/0,$$

where the penultimate one is the canonical injection. If we denote by f their composition, we deduce that the lattice $\overline{t}/0$ is isomorphic to its homomorphic image $f(\overline{t}/0)$, which is a sublattice of $L' := (s_2 \vee \cdots \vee s_n)/\,0$.

Denote $T' := \{f(t_2), \ldots, f(t_k)\}$ and $S' := \{s_2, \ldots, s_n\}$. Then, S' and T' satisfy the hypotheses from the statement of our theorem for the lattice L', so we can proceed by induction on n to obtain $k-1 \leqslant n-1$, and then, $k \leqslant n$, as desired. Observe that the first step for $n = 1$ of the induction is obviously true.

(2) By (1), we have $k \leqslant n$. Interchange now S with T and apply again (1) to obtain $n \leqslant k$. Then $n = k$, as desired. □

The statement and proof of Theorem 2.2.8 resemble the well-known *Steinitz Replacement Theorem* from Linear Algebra allowing to define an invariant of any vector space V, namely its *dimension* $\dim(V)$. Similarly, based on Theorem 2.2.8, we can associate with any lattice $L \in \mathcal{M}_0$ an invariant $u(L) \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, called its *Goldie* or *uniform dimension*.

Definition. *We say that a non-zero lattice $L \in \mathcal{M}_0$ has* Goldie dimension *or* uniform dimension *$n \in \mathbb{N}$, and we write $u(L) = n$, if there exists an independent subset S of L with $|S| = n$, and $\bigvee S \in E(L)$. By Theorem 2.2.8, $u(L)$ is well defined. If no such integer n exists, we write $u(L) = \infty$. For $L = 0$ we define $u(L) = 0$.* □

It is easy to check from the definition that $u(L) = 1 \Longleftrightarrow L$ is uniform.

For any ring R with identity and any unital right R-module M, we define the *Goldie dimension* of M, denoted $u(M)$, as being the Goldie dimension $u(\mathcal{L}(M))$ of the lattice $\mathcal{L}(M)$ of all its submodules. Similarly, we say that the module M has *finite length* if so is the lattice $\mathcal{L}(M)$, and $\ell(M)$ will denote its length.

Proposition 2.2.9. *A lattice $L \in \mathcal{M}_0$ has finite Goldie dimension if and only if L does not contain an infinite independent subset.*

Proof. If $u(L) = n < \infty$, then L contains no infinite independent subset by Theorem 2.2.8(1). Conversely, suppose that L does not contain an infinite independent subset. Then, by Corollary 2.2.5, there exists a finite independent set S of uniform elements of L such that $\bigvee S \in E(L)$. By definition, L has finite Goldie dimension. □

Corollary 2.2.10. *For any lattice $L \in \mathcal{M}_0$,*

$$u(L) = \sup\{\, k \,|\, L \text{ contains an independent subset of } k \text{ elements}\}.$$

Proof. Let $\lambda \leqslant \infty$ be this supremum. If $\lambda = \infty$, then necessarily $u(L) = \infty$ by Theorem 2.2.8(1). Now, if $\lambda < \infty$ then $u(L)$ must be finite by Proposition 2.2.9, and, again by Theorem 2.2.8(1), we deduce that $u(L) = \lambda$. □

Corollary 2.2.11. *The following statements hold for a lattice $L \in \mathcal{M}_0$.*

(1) *If L is Noetherian then $u(L) < \infty$.*
(2) *If $L \in \mathcal{M}_{0,1}$ has finite length, then $u(L) \leqslant \ell(L)$, with equality if and only if L is semi-atomic.*

Proof. (1) Assume that L is Noetherian but $u(L) = \infty$. Then, by Proposition 2.2.9, L contains an infinite independent subset $T = \{\, t_i \,|\, i \in \mathbb{N}\,\}$, and we may consider the ascending chain $t_1 < t_1 \vee t_2 < t_1 \vee t_2 \vee t_3 < \ldots$, which is a contradiction.

(2) Let $n := \ell(L)$. If L contains a finite independent set $T = \{t_1, \dots t_k\}$ for some $k \in \mathbb{N}$, then, by a induction on k and repeated application of Proposition 2.1.8 we have

$$k \leqslant \sum_{i=1}^{k} \ell(t_i/0) = \ell((t_1 \vee \cdots \vee t_k)/0) \leqslant \ell(L) = n.$$

By Corollary 2.2.10, we deduce that $u(L) \leqslant n$.

If L is semi-atomic, then $u(L) = n$ by the definition of the Goldie dimension, Conversely, if $u(L) = n$, then L contains a finite independent set $S = \{s_1, \dots, s_n\}$. Applying the inequalities above for S and $k = n$, we deduce that $s_1 \vee \cdots \vee s_n = 1$ and $\ell(s_i/0) = 1$, i.e., s_i is an atom for all $1 \leqslant i \leqslant n$. Thus, L is semi-atomic. $\square$

If V is now any finite dimensional vector space over a field F, then Corollary 2.2.11(2) gives for $L = \mathcal{L}(V)$ the equalities $\dim(V) = \ell(V) = u(V)$ relating the dimension, length, and Goldie dimension of the module V over the field F.

Note that Artinian lattices need not have finite Goldie dimension. Indeed, let F be any field, let V be an infinite dimensional vector space over F, and let L denote the sublattice of the lattice $\mathcal{L}(V)$ of all subspaces of V consisting of V and all finite dimensional subspaces of V. Then, it is clear that L is an Artinian lattice but L does not have finite Goldie dimension. For another example, let $\mathcal{L}(\mathbb{Z})$ be the lattice of all ideals of the ring $\mathbb{Z}$ of rational integers. Then, the Artinian lattice $\mathcal{L}(\mathbb{Z})^o$ does not have finite Goldie dimension because it contains the infinite independent set $\{\, p\mathbb{Z} \mid p \text{ prime} \,\}$. We shall see in Corollary 2.2.17 that an Artinian lattice has finite Goldie dimension if and only if it is E-complemented.

For any $\alpha \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ we make the usual convention that $\alpha + \infty = \infty$.

Corollary 2.2.12. *Let $L \in \mathcal{M}_0$, $a \in L$, $n \in \mathbb{N}$, and $\{a_1, \dots, a_n\}$ an independent subset of L. Then, the following statements hold.*

(1) $u((a_1 \vee \cdots \vee a_n)/0) = \sum_{1 \leqslant i \leqslant n} u(a_i/0)$.
(2) $u(a/0) \leqslant u(L)$ *with equality if* $a \in E(L)$.
(3) *If* $a \notin E(L)$, *then* $u(a/0) < u(L)$, *unless* $u(a/0) = u(L) = \infty$.
(4) *If* $u(L) < \infty$, *then* $a \in E(L) \iff u(L) = u(a/0)$.

Proof. (1) We proceed by induction on n. Clearly, by Lemma 2.2.1, it is sufficient to consider only the case $n = 2$. So, let a and b be non-zero elements of L such that $a \wedge b = 0$. Then, we are going to prove that

$$u((a \vee b)/0) = u(a/0) + u(b/0).$$

Suppose that $u(a/0) = \infty$. Then $a/0$ contains an infinite independent subset and hence so too does $(a \vee b)/0$, and the result is true in this case.

So, we may assume that $a/0$ and $b/0$ both have finite Goldie dimension. Then, there exist a finite independent subset S in $a/0$ and a finite independent subset T in $b/0$ with $\bigvee S \in E(a/0)$ and $\bigvee T \in E(b/0)$. Because $a \wedge b = 0$, $S \cup T$ is an independent subset of $(a \vee b)/0$ by Corollary 2.2.2, so, in particular $S \cap T = \varnothing$. Now, observe that $\bigvee(S \cup T) = \big(\bigvee S\big) \vee \big(\bigvee T\big) \in E((a \vee b)/0)$ by Lemma 1.1.7 and Proposition 1.2.9. By the definition of the Goldie dimension, we deduce the desired equality

$$u((a \vee b)/0) = |\, S \cup T \,| = |\, S \,| + |\, T \,| = u(a/0) + u(b/0).$$

(2) Assume first that $u(a/0) = \infty$. Then, by Proposition 2.2.9, $\infty = u(a/0) = u(L)$. So, we may assume that $u(a/0) < \infty$. The result is trivial if $L = 0$, so we may suppose that L is non-zero. If $a \in E(L)$, then, by the definition of the Goldie dimension, there exists a finite independent set S of uniform elements of $a/0$ such that $\bigvee S \in E(a/0)$. Lemma 1.2.1 gives that $\bigvee S \in E(L)$, and so, $|S| = u(a/0) = u(L)$.

(3) Assume that $a \notin E(L)$. If $u(a/0) = \infty$, then necessarily $u(L) = \infty$ too, so we may consider only the case $u(a/0) = n < \infty$. It follows that there exists an independent subset $\{s_1, \dots, s_n\}$ of $a/0$. Because $a \notin E(L)$, there exists $0 \neq b \in L$ such that $b \wedge a = 0$. Then $b \wedge (s_1 \vee \cdots \vee s_n) = 0$, so $\{b, s_1, \dots, s_n\}$ is an independent subset of L. Hence $u(L) \geqslant n + 1 > n = u(a/0)$ by Theorem 2.2.8(1), as desired.

(4) follows immediately from (2) and (3). □

Properties of the Goldie dimension

Theorem 2.2.13. *Let $a \leqslant b$ be elements of a lattice $L \in \mathcal{M}_0$. Then*

$$u(a/0) \leqslant u(b/0) \leqslant u(a/0) + u(b/a).$$

Proof. If $u(b/0) = \infty$ then $u(a/0) \leqslant u(b/0)$. On the other hand, if $u(b/0)$ is finite then $u(a/0) \leqslant u(b/0)$ by Theorem 2.2.8(1). Thus, in any case, $u(a/0) \leqslant u(b/0)$. Next, if $u(a/0) = \infty$ or $u(b/a) = \infty$ then clearly $u(b/0) \leqslant u(a/0) + u(b/a)$.

Thus suppose that $a/0$ and b/a both have finite Goldie dimension. By Corollary 2.2.5, there exists a finite independent subset S of uniform elements of $a/0$ such that $\bigvee S \in E(a/0)$. Moreover, by Lemma 2.2.4 there exists a finite independent subset T of uniform elements of $b/0$ such that $S \cup T$ is independent and $\bigvee(S \cup T) \in E(b/0)$, and then

$$u(b/0) = |S \cup T| = |S| + |T| = u(a/0) + |T|.$$

We complete the proof by showing that $|T| \leqslant u(b/a)$. Note first that $a \wedge \left(\bigvee T\right) \neq 0$ would imply that $(\bigvee S) \wedge (\bigvee T) \neq 0$, contradicting Corollary 2.2.2. Thus $a \wedge (\bigvee T) = 0$, and then $T \cup \{a\}$ is independent by Lemma 2.2.1. Now suppose that $T = \{t_1, \dots, t_n\}$ for some $n \in \mathbb{N}$. Then, for each $1 \leqslant i \leqslant n - 1$,

$$\begin{aligned}(t_{i+1} \vee a) \wedge ((t_1 \vee a) \vee \cdots \vee (t_i \vee a)) &= (t_{i+1} \vee a) \wedge (a \vee t_1 \vee \cdots \vee t_i) \\ &= a \vee ((t_{i+1} \vee a) \wedge (t_1 \vee \cdots \vee t_i)) = a \vee 0 = a,\end{aligned}$$

by Corollary 2.2.2. It follows that $\{t_1 \vee a, \dots, t_n \vee a\}$ is an independent subset of b/a by Lemma 2.2.1. Finally, Theorem 2.2.8(1) gives that $|T| = n \leqslant u(b/a)$, as required. □

Corollary 2.2.14. *Let a be any element of a lattice $L \in \mathcal{M}_0$. Then*

$$u(a/0) \leqslant u(L) \leqslant u(a/0) + u([a)).$$

Proof. Suppose first that L has a greatest element 1. Then Theorem 2.2.13 gives the following:

$$u(a/0) \leqslant u(1/0) = u(L) \leqslant u(a/0) + u(1/a) = u(a/0) + u([a)).$$

Now suppose that L does not have a greatest element. We define a new lattice L_1 by formally adjoining a greatest element i to L as follows. Given $a, b \in L$, the least upper bound of a and b in L_1 is $a \vee b$ in L, the greatest lower bound of a and b in L_1 is $a \wedge b$ in L, $a \vee i = i$, and $a \wedge i = a$. It is easy to check that L_1 is a modular lattice. By the first part of the proof we obtain the desired result

$$u(a/0) \leqslant u(L_1) = u(L) \leqslant u(a/0) + u(i/a) = u(a/0) + u([a)). \qquad \square$$

THEOREM 2.2.15. *Let a and b be any elements of a lattice $L \in \mathcal{M}_0$. Then*
$$u(a/0) + u(b/0) \leqslant u((a \vee b)/0) + u((a \wedge b)/0).$$

PROOF. If $a \wedge b = 0$ then the result follows from Corollary 2.2.12(1). Now we consider the general case. The result is clear when $u((a \vee b)/0) = \infty$. Thus, suppose that $u((a \vee b)/0) < \infty$. By Corollary 2.2.6, there exists $c \in b/0$ with $(a \wedge b) \wedge c = 0$ and $(a \wedge b) \vee c \in E(b/0)$. Corollary 2.2.12(1) and (4) give that $u(b/0) = u((a \wedge b)/0) + u(c/0)$. Note that $a \wedge c \leqslant a \wedge (b \wedge c) = 0$, so, using again Corollary 2.2.12(1) we have

$$u((a \vee b)/0) \geqslant u((a \vee c)/0) = u(a/0) + u(c/0) = u(a/0) + (u(b/0) - u((a \wedge b)/0)),$$

and the result follows. $\square$

Observe that if a lattice L has finite Goldie dimension, it may happen that not all of its sublattices $[a)$ are so. To see that, consider the lattice $K = \mathcal{L}(\mathbb{Q})$ of all subgroups of the Abelian group $\mathbb{Q}$ of rational numbers. Since $\mathbb{Q}$ is a uniform $\mathbb{Z}$-module, we have $u(K) = 1$, but the interval $[\mathbb{Z})$ of K has infinite Goldie dimension because the torsion Abelian group $\mathbb{Q}/\mathbb{Z}$ is the infinite direct sum of its primary components.

We say that a lattice L is QFD (acronym for *Q*uotient *F*inite *D*imensional) if $[a)$ has finite Goldie dimension for all $a \in L$. By Proposition 2.1.1 and Corollary 2.2.11(1), Noetherian lattices are QFD.

Lattices with finite Goldie dimension

We now consider lattices with finite Goldie dimension and derive some of their properties. We begin with different characterizations for a lattice to have finite Goldie dimension.

THEOREM 2.2.16. *The following statements are equivalent for a non-zero lattice $L \in \mathcal{M}_0$.*

(1) *L has finite Goldie dimension.*
(2) *Given any ascending chain $a_1 \leqslant a_2 \leqslant \dots$ of elements of L, there exists $n \in \mathbb{N}$ with $a_i \in E(a_{i+1}/0)$ for all $i \geqslant n$.*
(3) (a) *L is E-complemented, and*
    (b) *given any descending chain $b_1 \geqslant b_2 \geqslant \dots$ of elements of L, there exists $m \in \mathbb{N}$ such that $b_{i+1} \in E(b_i/0)$ for all $i \geqslant m$.*

PROOF. (1) $\Longrightarrow$ (2) Suppose that L has finite Goldie dimension. By Theorem 2.2.13, for any ascending chain $a_1 \leqslant a_2 \leqslant \dots$ in L, we have

$$u(a_1/0) \leqslant u(a_2/0) \leqslant \dots \leqslant u(L),$$

and hence there exists $n \in \mathbb{N}$ such that

$$u(a_n/0) = u(a_{n+1}/0) = \dots.$$

By Corollary 2.2.12(4), $a_i \in E(a_{i+1}/0)$ for all $i \geqslant n$.

(2) $\Longrightarrow$ (1) Suppose that L contains an infinite independent subset $\{c_1, c_2, \dots\}$. Then

$$c_1 < c_1 \vee c_2 < c_1 \vee c_2 \vee c_3 < \dots,$$

and $c_1 \vee \dots \vee c_n \notin E((c_1 \vee \dots \vee c_{n+1})/0)$ for all $n \geqslant 1$ by Corollary 2.2.12(3).

(1) $\implies$ (3) Suppose that L has finite Goldie dimension. By Corollary 2.2.6, L satisfies (3)(a). Also L satisfies (3)(b) by adapting the proof of (1) $\implies$ (2).

(3) $\implies$ (1) Assume that L is E-complemented but L does not have finite Goldie dimension. Suppose that $L \setminus \{0\} = E(L)$. Then, because $L \neq 0$, it has a non-zero element which is necessarily uniform and essential in L, so $u(L) = 1$. Therefore $L \setminus \{0\} \neq E(L)$. Choose $0 \neq x \in L \setminus E(L)$. By hypothesis, there exists a non-zero element $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in E(L)$. If $x/0$ and $y/0$ both have finite uniform dimension then so too does $(x \vee y)/0$ by Corollary 2.2.12(1), and consequently, also L has finite Goldie dimension by Corollary 2.2.12(4). Thus, there exists $b_1 \in L \setminus E(L)$ such that $b_1/0$ does not have finite Goldie dimension. Moreover, Lemma 1.2.12(2) gives that $b_1/0$ is E-complemented. Repeating this argument we obtain an element $b_2 \in (b_1/0) \setminus E(b_1/0)$ such that $b_2/0$ does not have finite Goldie dimension. Proceed in the same way to obtain a descending chain $b_1 \geqslant b_2 \geqslant \dots$ in L such that $b_{m+1} \notin E(b_m/0)$ for all $m \geqslant 1$. Now (3) $\implies$ (1) follows immediately. $\square$

Corollary 2.2.17. *An Artinian lattice $L \in \mathcal{M}_0$ has finite Goldie dimension $\iff$ L is E-complemented.*

Proof. Apply Corollary 2.2.6 and Theorem 2.2.16. $\square$

Next we consider closed elements once more. First we prove the following straightforward lemma.

Lemma 2.2.18. *Let c be a non-zero closed element in a lattice $L \in \mathcal{M}_0$ and let $\{a_1, \dots, a_n\}$ be an independent set of (non-zero elements) of $[c)$. Then, there exist elements $0 \neq b_i \in a_i/0\,(1 \leqslant i \leqslant n)$ such that $\{c, b_1, \dots, b_n\}$ is an independent subset of L.*

Proof. For each $1 \leqslant i \leqslant n$, $c < a_i$ but $c \notin E(a_i/0)$ and hence $c \wedge b_i = 0$ for some $0 \neq b_i \in a_i/0$. It follows that

$$b_{j+1} \wedge (c \vee b_1 \vee \dots \vee b_j) \leqslant a_{j+1} \wedge (a_1 \vee \dots \vee a_j) = c$$

for each $1 \leqslant j \leqslant n-1$, so that

$$b_{j+1} \wedge (c \vee b_1 \vee \dots \vee b_j) \leqslant c \wedge b_{j+1} = 0.$$

By Lemma 2.2.1, the set $\{c, b_1, \dots, b_n\}$ is independent. $\square$

Theorem 2.2.19. *The following statements hold for a lattice $L \in \mathcal{M}_0$ with finite Goldie dimension and $c \in L$.*

(1) *If $c \in C(L)$ then $u(L) = u(c/0) + u([c))$.*
(2) *If L is essentially closed and $u(L) = u(c/0) + u([c))$ then $c \in C(L)$.*

Proof. (1) Suppose that $c \in C(L)$. If $u([c)) = \infty$ then $[c)$ contains an infinite independent subset so that, by Lemma 2.2.18, L contains an infinite independent subset and $u(L) = \infty$, a contradiction. Thus $[c)$ has finite Goldie dimension. By Corollary 2.2.5 there exist $n \in \mathbb{N}$ and a set $\{u_1, \dots, u_n\}$ of uniform elements of $[c)$ such that $u_1 \vee \dots \vee u_n \in E([c))$. By Lemma 2.2.18 and its proof, for each $1 \leqslant i \leqslant n$ there exists $0 \neq w_i \in u_i/0$ such that $c \wedge w_i = 0$ and $\{c, w_1, \dots, w_n\}$ is an independent set of elements of L. Now, Proposition 1.1.4 gives

$$w_i/0 \simeq w_i/(w_i \wedge c) \simeq (w_i \vee c)/c \leqslant u_i/c,$$

so that w_i is a uniform element for each $1 \leqslant i \leqslant n$. Set $w := w_1 \vee \cdots \vee w_n$. Note that $c \wedge w = 0$. We show next that $c \vee w \in E(L)$. Firstly, $c \vee w = (c \vee w_1) \vee \cdots \vee (c \vee w_n)$ and $c \neq c \vee w_i \leqslant u_i \, (1 \leqslant i \leqslant n)$. By Lemma 2.2.7, $c \vee w \in E([c))$, so $c \vee w \in E(L)$ by Lemma 1.2.10. By Corollary 2.2.12,

$$u(L) = u((c \vee w)/0) = u(c/0) + u(w/0) = u(c/0) + n = u(c/0) + u([c)).$$

(2) Suppose that L is essentially closed and let $c \in L$ be such that

$$u(L) = u(c/0) + u([c)).$$

Proposition 1.2.22 gives that $c \in E(d/0)$ for some $c \leqslant d$ with $d \in C(L)$. Applying (1) and Corollary 2.2.12(4), we have

$$u(c/0) + u([c)) = u(L) = u(d/0) + u([d)) = u(c/0) + u([d)),$$

and so $u([c)) = u([d))$. But, using (1) again, $d \in C([c))$ implies that

$$u([d)) = u([c)) = u([d)) + u(d/c),$$

and hence $u(d/c) = 0$. By Lemma 2.2.3, we deduce that $c = d$, which proves that $c \in C(L)$. □

Corollary 2.2.20. *Let $L \in \mathcal{M}_0$ be a lattice with finite Goldie dimension, and let $c < b$ in L with $c \in C(L)$. Then $u(c/0) < u(b/0)$.*

Proof. Note first that $c \in C(b/0)$ and hence $u(b/0) = u(c/0) + u(b/c)$ by Theorem 2.2.19. But $b/c \neq 0$ implies that $u(b/c) \neq 0$ and the result follows. □

Corollary 2.2.21. *Every lattice $L \in \mathcal{M}_0$ with finite Goldie dimension satisfies* ACC *and* DCC *on closed elements.*

Proof. Apply Corollary 2.2.20. □

Note that Corollary 2.2.21 implies that every lattice with finite Goldie dimension satisfies ACC and DCC on pseudo-complements (see Proposition 1.2.16). Our next objective is to present partial converses of Corollary 2.2.21. Recall that we have denoted by $P(L)$ the set of all pseudo-complement elements of L.

Theorem 2.2.22. *The following statements are equivalent for an essentially closed lattice $L \in \mathcal{M}_0$.*

(1) *L has finite Goldie dimension.*
(2) *$C(L)$ is a Noetherian poset.*
(3) *L is E-complemented and $P(L)$ is a Noetherian poset.*

Proof. (1) $\Longrightarrow$ (3) Apply Corollaries 2.2.6 and 2.2.21.

(3) $\Longrightarrow$ (2) Apply Corollary 1.2.17.

(2) $\Longrightarrow$ (1) Suppose that L does not have finite Goldie dimension and let $\{a_1, a_2, \dots\}$ be an infinite independent subset of L. By Proposition 1.2.22, there exists $c_1 \in C(L)$ such that $a_1 \in E(c_1/0)$. Proposition 1.2.22 gives an element $c_2 \in C(L)$ such that $c_1 \vee a_2 \in E(c_2/0)$. Then there exists $c_3 \in C(L)$ such that $c_2 \vee a_3 \in E(c_3/0)$, and so on. This produces a chain $c_1 \leqslant c_2 \leqslant \dots$ of closed elements of L. Next, $c_1 \wedge a_2 = 0$ because $(c_1 \wedge a_2) \wedge a_1 \leqslant a_1 \wedge a_2 = 0$ and $a_1 \in E(c_1/0)$. By Lemma 1.2.1 and Proposition 1.2.9, we deduce that $a_1 \vee a_2 \in E(c_2/0)$. In the same way $a_1 \vee a_2 \vee a_3 \in E(c_3/0)$ and, in

general, $a_1 \vee \cdots \vee a_n \in E(c_n/0)$ for all $n \in \mathbb{N}$. If $c_n = c_{n+1}$ for some $n \in \mathbb{N}$, then it would follow $a_{n+1} \leqslant c_{n+1} = c_n$, and so $a_{n+1} \wedge (a_1 \vee \cdots \vee a_n) \neq 0$, a contradiction. Thus $c_1 < c_2 < \dots$ and L does not satisfy ACC on closed elements. □

We now consider descending chain conditions.

THEOREM 2.2.23. *The following statements are equivalent for a pseudo-complemented lattice $L \in \mathcal{M}_0$.*

(1) *L has finite Goldie dimension.*
(2) *$C(L)$ is an Artinian poset.*
(3) *$P(L)$ is an Artinian poset.*

PROOF. (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) by Corollaries 2.2.21 and 1.2.17 and Proposition 1.2.18(1).

(2) $\Longrightarrow$ (1) Note that $P(L) = C(L)$ and $P(b/0) = C(b/0)$ for all $b \in L$ by Lemma 1.2.20, Corollary 1.2.17, and Proposition 1.2.18. Suppose that L does not have finite Goldie dimension and let $\{a_1, a_2, \dots\}$ be an infinite independent subset of L. Let b_1 be any pseudo-complement of a_1 in L. Note that $b_1 \in C(L)$ by Proposition 1.2.16. By Lemma 1.2.15, $a_1 \vee b_1 \in E(L)$ so that $(a_1 \vee b_1) \wedge a_2 \neq 0$. Lemma 1.2.6 gives that $(a_1 \vee a_2) \wedge b_1 \neq 0$. By Lemma 1.2.20 there exists a pseudo-complement b_2 of $(a_1 \vee a_2) \wedge b_1$ in $b_1/0$. Then $b_1 > b_2$, for otherwise we would have $b_1 = b_2$, and then $0 = (a_1 \vee a_2) \wedge b_1 \wedge b_2 = (a_1 \vee a_2) \wedge b_1$, a contradiction. By Corollary 1.2.14, $b_2 \in C(L)$. Next, $((a_1 \vee a_2) \wedge b_1) \vee b_2 \in E(b_1/0)$ by Lemma 1.2.15, and hence

$$a_1 \vee ((a_1 \vee a_2) \wedge b_1) \vee b_2) \in E(L),$$

by Lemmas 1.2.1 and 1.2.7. Next, using Lemma 1.2.1 again we see that $a_1 \vee a_2 \vee b_2 \in E(L)$ because

$$a_1 \vee ((a_1 \vee a_2) \wedge b_1) \vee b_2 \leqslant a_1 \vee a_2 \vee b_2.$$

In particular, this means that $(a_1 \vee a_2 \vee b_2) \wedge a_3 \neq 0$ and hence $(a_1 \vee a_2 \vee a_3) \wedge b_2 \neq 0$ by Lemma 1.2.6. Let b_3 be a pseudo-complement of $(a_1 \vee a_2 \vee a_3) \wedge b_2$ in $b_2/0$ (Lemma 1.2.20). Note that $b_2 > b_3$ and, by Corollary 1.2.14 and Proposition 1.2.16, $b_3 \in C(L)$. Repeat this argument to produce an infinite descending chain $b_1 > b_2 > b_3 > \dots$ of closed elements b_i $(i \geqslant 1)$ of L. Thus L does not satisfy DCC on closed elements, and we are done. □

COROLLARY 2.2.24. *The following statements are equivalent for a strongly pseudo-complemented lattice $L \in \mathcal{M}_0$.*

(1) *L has finite Goldie dimension.*
(2) *$C(L)$ is a Noetherian poset.*
(3) *$P(L)$ is a Noetherian poset.*
(4) *$C(L)$ is an Artinian poset.*
(5) *$P(L)$ is an Artinian poset.*

PROOF. By Theorems 1.2.24, 2.2.22, and 2.2.23. □

Irreducible decompositions and Goldie dimension

Now we consider another type of special elements of a lattice L and we show how these elements can be used to characterize when the lattice has finite Goldie dimension.

An element x of a lattice L is said to be *meet irreducible*, or just *irreducible,* provided whenever $x = y \wedge z$ for some $y, z \in L$ then $x = y$ or $x = z$.

The concept of an irreducible element of a lattice is the latticial counterpart of that of an *irreducible submodule* of a module. Notice that an irreducible submodule of a module M is by definition a proper submodule of M. For this reason, if a lattice L has a greatest element 1, then, an irreducible element of L has to be, by definition, a proper element of L; in this case, clearly $x \in L$ is an irreducible element of L if and only if $1/x$ is a uniform lattice.

THEOREM 2.2.25. *Let $L \in \mathcal{M}_0$ be a non-zero essentially closed lattice with finite Goldie dimension. Then there exist $n \in \mathbb{N}$ and irreducible elements $a_i\,(1 \leqslant i \leqslant n)$ of L such that $0 = a_1 \wedge \cdots \wedge a_n$.*

PROOF. Suppose that $u(L) = n$, for some $n \in \mathbb{N}$. By definition, there exists an independent set of uniform elements $u_i \in L\,(1 \leqslant i \leqslant n)$ such that $u_1 \vee \cdots \vee u_n \in E(L)$. For each $1 \leqslant i \leqslant n$, Proposition 1.2.22 gives a closed element c_i in L such that

$$u_1 \vee \cdots \vee u_{i-1} \vee u_{i+1} \vee \cdots \vee u_n \in E(c_i/0).$$

Note that $c_i \wedge u_i = 0$ and $u(c_i/0) = n - 1$ for every $1 \leqslant i \leqslant n$). Let $c = c_1 \wedge \cdots \wedge c_n$. Then $c \wedge u_1 \leqslant c_1 \wedge u_1 = 0$, so that $c \wedge u_1 = 0$. Now suppose that $c \wedge (u_1 \vee \cdots \vee u_i) = 0$ for some $1 \leqslant i \leqslant n - 1$. Then

$$(c \vee (u_1 \vee \cdots \vee u_i)) \wedge u_{i+1} \leqslant c_{i+1} \wedge u_{i+1} = 0.$$

By Lemma 1.2.6, $c \wedge (u_1 \vee \cdots \vee u_{i+1}) = 0$. By induction, we have $c \wedge (u_1 \vee \cdots \vee u_n) = 0$, and hence $c = 0$ because $u_1 \vee \cdots \vee u_n \in E(L)$.

Let $1 \leqslant j \leqslant n$. By Theorem 2.2.19,

$$n = u(L) = u(c_j/0) + u([c_j)) = n - 1 + u([c_j)),$$

so that $u([c_j)) = 1$. It is easy to see that this implies that c_j is an irreducible element of L. Thus, c_j is an irreducible element of L for each $1 \leqslant j \leqslant n$, as required. $\square$

Clearly, 0 is a closed element in any lattice. The next result shows that the conclusion of Theorem 2.2.25 holds for any closed element of L and not only for 0.

COROLLARY 2.2.26. *Let $L \in \mathcal{M}_0$ be a non-zero essentially closed lattice with finite Goldie dimension. Then every closed element c of L can be written as $c = a_1 \wedge \cdots \wedge a_m$ for some $m \in \mathbb{N}$ and irreducible elements $a_i\,(1 \leqslant i \leqslant m)$ of L.*

PROOF. Let $c \in C(L)$. By Corollary 1.2.23 the sublattice $[c)$ of L is essentially closed and, by Theorem 2.2.19(1), $[c)$ has finite Goldie dimension. The result follows by Theorem 2.2.25. $\square$

The two results above can be reformulated by using the concept of an *irreducible decomposition.* If x is an element of a lattice L, then a representation

$$x = x_1 \wedge \ldots \wedge x_n$$

of x as a meet of finitely many irreducible elements $x_1, \ldots, x_n$ of L is called a finite *irreducible decomposition* of x, which is said to be *irredundant* if for each $1 \leqslant i \leqslant n$,

$$x \neq x_1 \wedge \ldots x_{i-1} \wedge x_{i+1} \wedge \ldots \wedge x_n.$$

If x has a finite irreducible decomposition, then clearly it has an irredundant finite irreducible decomposition by deleting in the original decomposition those irreducible elements x_i such that $\bigwedge_{j\neq i} x_j \leqslant x_i$.

THEOREM 2.2.27. *Let $L \in \mathcal{M}_0$ be a non-zero lattice such that 0 has a finite irreducible decomposition $0 = b_1 \wedge \cdots \wedge b_n$ for some $m \in \mathbb{N}$. Then L has finite uniform dimension and $u(L) \leqslant n$. Moreover, $u(L) = n$ if and only if the considered irreducible decomposition is irredundant.*

PROOF. As just mentioned above, the given irreducible decomposition of 0 produces an irredundant irreducible decomposition $0 = a_1 \wedge \cdots \wedge a_m$, with $1 \leqslant m \leqslant n$. For each $1 \leqslant i \leqslant m$ set

$$w_i := a_1 \wedge \cdots \wedge a_{i-1} \wedge a_{i+1} \wedge \ldots \wedge a_m.$$

Since $a_i \wedge w_i = 0$ for each $1 \leqslant i \leqslant m$, we deduce that $\{w_1, \ldots, w_m\}$ is an independent subset of L. Moreover, for each $1 \leqslant i \leqslant m$,

$$w_i/0 = w_i/(w_i \wedge a_i) \simeq (w_i \vee a_i)/a_i.$$

It follows that w_i is uniform for each $1 \leqslant i \leqslant m$ because a_i is irreducible in $(w_i \vee a_i)/a_i$.

We are now going to prove by induction on m that $\bigvee_{1\leqslant i\leqslant m} w_i \in E(L)$, which will imply that $u(L) = m \leqslant n$ by Theorem 2.2.8(1), as desired. Remember that m is a positive integer such that 0 has a finite irredundant irreducible decomposition $0 = a_1 \wedge \cdots \wedge a_m$.

If $m = 1$ then $0 = a_1$ is an irreducible element of L, i.e., L is a uniform lattice, which means exactly that $u(L) = 1$. If $m \geqslant 2$, then observe that $w_1 = a_2 \wedge \ldots \wedge a_m$ is an irredundant decomposition of the least element w_1 in the lattice $[w_1)$, and so $u([w_1)) = m - 1$.

Assume that $w_1 \vee \cdots \vee w_m \notin E(L)$. Then $b \wedge (w_1 \vee \cdots \vee w_m) = 0$ for some non-zero $b \in L$. By Lemma 2.2.1, we deduce that $\{w_1, \ldots w_m, b\}$ is an independent subset of L, so, by Proposition 1.1.4, we have

$$(w_2 \vee \cdots \vee w_m \vee b)/0 = (w_2 \vee \cdots \vee w_m \vee b)/(w_1 \wedge (w_2 \vee \cdots \vee w_m \vee b)) \simeq$$
$$\simeq (w_1 \vee w_2 \vee \cdots \vee w_m \vee b)/w_1 \subseteq [w_1).$$

By Theorem 2.2.8(1), it follows that $u([w_1)) \geqslant m \geqslant 2$. We obtained a contradiction because we have seen that w_i is a uniform lattice, i.e., $u([w_i)) = 1$ for each $1 \leqslant i \leqslant m$, in particular $u([w_1)) = 1$. Consequently, we must have $w_1 \vee \cdots \vee w_m \notin E(L)$, as desired.

Conversely, suppose that $u(L) = n$. If the given irreducible decomposition

$$0 = b_1 \wedge \cdots \wedge b_n$$

is not irredundant, then as above, it produces an irredundant irreducible decomposition

$$0 = a_1 \wedge \cdots \wedge a_m$$

with $m < n$, and then from (1) it follows that $u(L) = m$, which is a contradiction. The result follows. □

COROLLARY 2.2.28. *An essentially closed lattice $L \in \mathcal{M}_0$ has finite Goldie dimension if and only if 0 has a finite irreducible decomposition in L.*

PROOF. The result follows from Theorems 2.2.25 and 2.2.27. □

By considering the opposite lattice L^o of a lattice L, all the results in this section can be dualized. An element x of a lattice L with greatest element 1 will be called *join irreducible* if $x \neq 1$, and whenever $x = y \vee z$ for some elements $y, z \in L$ then $x = y$ or $x = z$. Thus, x is join irreducible in L if and only if x is irreducible in L^o. Corollary 2.2.28 gives the following fact:

A superfluously closed modular lattice L with greatest element 1 has finite dual Goldie dimension if and only if there exist $n \in \mathbb{N}$ and join irreducible elements a_i $(1 \leqslant i \leqslant n)$ such that $1 = a_1 \vee \cdots \vee a_n$.

Other dual results can be derived by the interested reader.

Subdirect irreducibility and completely irreducible elements

The concept of *subdirectly irreducible*, abbreviated *SI*, appears in various circumstances: universal algebras, rings, modules, lattices, posets, etc. Loosely speaking, an object of a category with direct products is called *subdirectly irreducible* if it cannot be represented as a subdirect product of "smaller" subobjects (i.e., proper epimorphic images). We shall illustrate below more precisely this concept for module categories.

A module M_R is called *subdirectly irreducible* if any representation of M as a *subdirect product* of other modules is trivial, i.e., for every family $(M_i)_{i \in I}$ of right R-modules and for every monomorphism $\varepsilon : M \rightarrowtail \prod_{i \in I} M_i$ such that $\pi_j \circ \varepsilon$ is an epimorphism $\forall\, j \in I$, $\exists\, i \in I$ such that $\pi_i \circ \varepsilon$ is an isomorphism, where $\pi_j : \prod_{i \in I} M_i \twoheadrightarrow M_j$, $j \in I$, are the canonical projections.

The concept of a subdirectly irreducible module turns out to be the dual of that of a cyclic module, and therefore such a module is also called *cocyclic* (see, e.g., [**88**]). A simple result states that a module M_R is subdirectly irreducible if and only if the poset $\mathcal{L}(M) \setminus \{0\}$ of non-zero submodules of M, ordered by inclusion, has a least element (see, e.g., [**88**, 14.8]). This naturally leads to the most general concept of a subdirectly irreducible poset.

A poset P with least element 0 is said to be *subdirectly irreducible*, abbreviated *SI,* if $P \neq \{0\}$ and the set $P \setminus \{0\}$ has a least element, i.e., there exists an element $0 \neq x_0 \in P$ such that $x_0 \leqslant x$ for every $0 \neq x \in P$. An element $s \in P$ is said to be a *subdirectly irreducible element of* P if the interval $1/s$ is a subdirectly irreducible poset, and the set of all subdirectly irreducible elements of P will be denoted by $\mathcal{S}(P)$.

Let L be a complete lattice. We say that L is *completely uniform* if $L \neq \{0\}$ and $\bigwedge_{i \in I} x_i \neq 0$ for any non-empty family $(x_i)_{i \in I}$ of non-zero elements $x_i \in L$. Notice that this concept does not agree with that of a *completely* $\mathbb{P}$ *poset* we introduced in Section 1.2.

An element $x \in L$ is said to be *completely irreducible*, abbreviated *CI,* if $x \neq 1$ and whenever $x = \bigwedge_{i \in I} a_i$ for a non-empty family $(a_i)_{i \in I}$ of elements of L, then $x = a_j$ for some $j \in I$, or shortly, if $x \neq 1$ and whenever $x = \bigwedge S$ for some $\varnothing \neq S \subseteq L$, then necessarily $x \in S$.

For any lattice L we denote by $\mathcal{I}(L)$ the set of all irreducible elements of L and by $\mathcal{I}^c(L)$ the set of all completely irreducible elements of L. For any module M_R we set $\mathcal{I}(M_R) := \mathcal{I}(\mathcal{L}(M_R))$ and $\mathcal{I}^c(M_R) := \mathcal{I}^c(\mathcal{L}(M_R))$.

Remarks 2.2.29. (1) The set $\mathcal{I}^c(L)$ may be empty: take as L the interval $[0, 1]$ of real numbers. However, for any non-zero module M_R we have $\mathcal{I}^c(M_R) \neq \varnothing$ (see, e.g., [**27**, Lemma 0.2]).

(2) Clearly, for any lattice L we have $\mathcal{I}^c(L) \subseteq \mathcal{I}(L)$. In general, the inclusion $\mathcal{I}^c(L) \subseteq \mathcal{I}(L)$ may be strict. Indeed, if L is the interval $[0,1]$ of the set $\mathbb{R}$ of real numbers, then $\mathcal{I}^c(L) = \varnothing$ and $\mathcal{I}(L) = L$.

(3) The set $\mathcal{I}(L)$ may be also empty. Indeed, let $I = [0,1) \subseteq \mathbb{R}$ and consider the lattice $L = (I \times I) \cup \{(1,1)\}$ ordered componentwise. For any $(a,b) \in L \setminus \{(1,1)\}$ we have $(a,b) = ((1+a)/2, b) \wedge (a, (1+b)/2)$. Hence (a,b) is not meet irreducible in L. However, for any non-zero module M_R we have $\mathcal{I}(M_R) \neq \varnothing$ by (1) and (2).

(4) If L is a complete lattice, then clearly $s \in L$ is a *subdirectly element of* L if and only if s is *completely irreducible*, so $\mathcal{S}(L) = \mathcal{I}^c(L)$. In the sequel, for the term of subdirectly irreducible element of any lattice, we will always use the term of completely irreducible element. □

PROPOSITION 2.2.30. *The following statements are equivalent for a non-zero complete lattice* L*.*

(1) L *is subdirectly irreducible.*
(2) $\bigwedge_{x \in L \setminus \{0\}} x \neq 0$.
(3) L *is completely uniform.*
(4) L *has an atom* a *such that* $a \leqslant x, \forall x \in L \setminus \{0\}$.
(5) L *has an atom* a *that is essential in* L.
(6) L *is uniform and* $\mathrm{Soc}(L) \neq 0$.

PROOF. (1) $\Longrightarrow$ (2) If x_0 is the least element of $L \setminus \{0\}$, then $0 \neq x_0 \leqslant \bigwedge_{x \in L \setminus \{0\}} x$.

(2) $\Longleftrightarrow$ (3) is clear.

(3) $\Longrightarrow$ (4) If $a := \bigwedge_{x \in L \setminus \{0\}} x$, then a is an atom, and $a \leqslant x, \forall x \in L \setminus \{0\}$.

(4) $\Longrightarrow$ (5) For any $x \in L \setminus \{0\}$, we have $a \leqslant x$, so $x \wedge a = a \neq 0$, i.e., a is essential in L.

(5) $\Longrightarrow$ (6) Let $x, y \in L \setminus \{0\}$. Then $a \wedge x \neq 0$ and $a \wedge y \neq 0$ since a is essential in L, so $a \wedge x = a$ and $a \wedge y = a$ since a is an atom of L. It follows that $a \leqslant x$ and $a \leqslant y$, which implies that $0 \neq a \leqslant x \wedge y$, i.e., L is uniform. Clearly $a \leqslant \mathrm{Soc}(L)$, so $\mathrm{Soc}(L) \neq 0$.

(6) $\Longrightarrow$ (1) Since $\mathrm{Soc}(L) \neq 0$, L has at least an atom, say a. We claim that a is the single atom of L; indeed, if a' is another atom of L, then $a \wedge a' \neq 0$ since L is uniform, so $a \wedge a' = a = a'$. It follows that $\mathrm{Soc}(L) = \{a\}$. For every $x \in L \setminus \{0\}$ we have $0 \neq x \wedge a \leqslant a$, so $x \wedge a = a$, and then $a \leqslant x$. This shows that a is the least element of $L \setminus \{0\}$; hence L is subdirectly irreducible. □

COROLLARY 2.2.31. *If* L *is a complete semi-Artinian lattice then* $\mathcal{I}(L) = \mathcal{I}^c(L)$.

PROOF. If $x \in \mathcal{I}(L)$, then the lattice $1/x$ is uniform, and $\mathrm{Soc}(1/x) \neq 0$ since L is semi-Artinian lattice, so $1/x$ is a subdirectly irreducible lattice by Proposition 2.2.30, i.e., $x \in \mathcal{I}^c(L)$, as desired. □

PROPOSITION 2.2.32. *The following statements are equivalent for a complete lattice* L *and* $1 \neq c \in L$.

(1) $c \in \mathcal{I}^c(L)$.
(2) *There exists* $0 \neq x_0 \in L$ *such that* c *is maximal with respect to* $x_0 \nleqslant c$.

PROOF. (1) $\Longrightarrow$ (2) Because the lattice L was assumed to be complete, we can consider its element $y_0 := \bigwedge_{x>c} x$. Then $y_0 > c$ since c is CI, so $y_0 \nleqslant c$. Now, c is maximal with respect to $y_0 \nleqslant c$, since for any $d > c$ we have $y_0 \leqslant d$ by the definition of y_0. Thus (2) holds with x_0 as y_0.

(2) $\Longrightarrow$ (1) If we consider again the element $y_0 := \bigwedge_{x>c} x$ of L, then clearly $y_0 \geqslant c$. In order to prove that c is CI, i.e., $1/c$ is a SI lattice, by Proposition 2.2.32, we must show that $y_0 \neq c$. We have $x_0 \leqslant x$ for every $x > c$ by the maximality condition of c in (2). This implies that $x_0 \leqslant y_0$. We cannot have $y_0 = c$ since then $x_0 \leqslant c$, which contradicts (2). $\square$

Observe that if $x_0 \neq 0$ is any compact element of a complete lattice L, then the set $L_0 = \{\, x \in L \,|\, x_0 \nleqslant x \,\}$ is inductive, so Zorn's Lemma can be applied to find a maximal element c of L_0, which is completely irreducible by Proposition 2.2.32.

Note that for any module M_R, the lattice $\mathcal{L}(M)$ of all submodules of M has the property that for each $N < P$ in $\mathcal{L}(M)$, the quotient module P/N has a subdirectly quotient module P/Q by Remarks 2.2.29(1), so we may say that the lattice $\mathcal{L}(M)$ is "rich in subdirectly irreducibles". We take this property as definition for an arbitrary lattice or poset.

A lattice L is said to be *rich in subdirectly irreducibles*, abbreviated *RSI*, if for every $a < b$ in L, the interval b/a has a subdirectly irreducible quotient interval b/c, where $a \leqslant c \leqslant b$. This concept is indispensable in the evaluation of the dual Krull dimension of lattices, see Section 2.3.

The next characterization of RSI lattices and its immediate corollary provides large classes of such lattices (see Section 2.3 for Gabriel and Krull dimension).

PROPOSITION 2.2.33. *The following assertions are equivalent for an upper continuous modular lattice L.*

(1) *L is RSI.*
(2) *For each $a < b$ in L there exist $x < y$ in b/a such that y/x is simple.*
(3) *For each $a < b$ in L there exist $x < y$ in b/a such that y/x is compact.*
(4) *For each $a < b$ in L there exist $x < y$ in b/a such that y/x is compactly generated.*
(5) *For each $a < b$ in L there exist $x < y$ in b/a such that y/x has (dual) Krull dimension.*
(6) *For each $a < b$ in L there exist $x < y$ in b/a such that y/x has Gabriel dimension.*

PROOF. See [**20**, Proposition 1.2]. $\square$

COROLLARY 2.2.34. *Let L be an upper continuous modular lattice. If L has Gabriel dimension or is compactly generated, then L is RSI. In particular, if L is Artinian, semi-Artinian, Noetherian, or has (dual) Krull dimension, then L is RSI.* $\square$

The property of a lattice L being RSI is related to the property of L being a lattice with *completely irreducible decomposition,*, abbreviated *CID*, which means that every $1 \neq a \in L$ can be written as a meet of a family, not necessarily finite, of CI elements of L, or equivalently $a = \bigwedge_{x \in \mathfrak{I}^c(1/a)} x$.

PROPOSITION 2.2.35. *The following statements are equivalent for a non-zero upper continuous modular lattice L.*

(1) *The lattice* L *is* RSI.

(2) *For every* $a < b$ *in* L*, one has* $a = \bigwedge_{x \in \mathcal{I}^c(b/a)} x$*.*

In particular, if L *is* RSI*, then* $0 = \bigwedge_{x \in \mathcal{I}^c(L)} x$*.*

PROOF. See [20, Lemma 1.6] □

REMARKS 2.2.36. (1) Consider the subset $L = \{0\} \cup [1/2, 1]$ of $\mathbb{R}$. Then 0 is the only CI element of the lattice L, but for every $0 < a < b$ in L, the interval $[a, b]$ has no CI elements; in particular, the lattice L is not RSI. This example shows that an upper continuous modular lattice L may satisfy the property $0 = \bigwedge_{x \in \mathcal{I}^c(L)} x$ of Proposition 2.2.35 without being necessarily RSI.

(2) By Corollary 2.2.34 and Proposition 2.2.35, any proper submodule of any module M_R is an intersection of CI submodules of M.

(3) Proposition 2.2.35 can be expressed by saying that a lattice L is RSI if and only if, for every $0 \neq b \in L$, the lattice $b/0$ is with CID. In particular, any RSI lattice is a lattice with CID. The converse may be not true. Indeed, consider the following example: let $L := \{\,(x, y) \,|\, x,\, y \in [0, 1],\ x + y \leqslant 1\} \cup \{(1, 1)\}$, where $[0, 1]$ is the unit interval in the set $\mathbb{R}$ of all real numbers. Then L, ordered componentwise by the usual relation $\leqslant$ is a complete semimodular lattice. Since the only covering pairs in L are $(x, 1 - x) \prec (1, 1)$, $x \in [0, 1]$, it follows that L is not RSI. However, every element of L can be written as an irredundant intersection of at most two coatoms, so, L is a lattice with CID. By *coatom* of L we mean an atom of L^o, i.e., a maximal element of $L \setminus \{1\}$.

(4) Observe that if L is a lattice with CID, and $a,\, b \in L$, then $b \nleqslant a$ if and only if there exists a CI element $c \in L$ such that $a \leqslant c$ and $b \nleqslant c$. Indeed, assume that for any CI element $c \in L$ such that $a \leqslant c$ we also have $b \leqslant c$. Since L is with CID, we can write $a = \bigwedge_{i \in I} c_i$ with CI irreducible elements c_i, $i \in I$. Then $b \leqslant c_i$, $\forall\, i \in I$, so $b \leqslant \bigwedge_{i \in I} c_i = a$. This shows the nontrivial implication. □

(5) In [9] it is investigated when a decomposition, not necessarily finite, of an element of an upper continuous modular lattice L with CID as a meet of irreducible/completely irreducible elements is irredundant or unique. Recall that a decomposition $x = \bigwedge_{i \in I} x_i$ is said to be an *irredundant irreducible decomposition*, respectively, an *irredundant completely irreducible decomposition*, if $\bigwedge_{j \in I \setminus \{i\}} x_j \nleqslant x_i$ for every $i \in I$, in other words, none of the x_i's can be omitted without changing the intersection. □

Recent results on completely irreducible submodules of a module and their connections with primal submodules, primary submodules, and their meet decompositions may be found in [33] and [35].

Dual Goldie dimension

We end this section by considering lattices L with a greatest element 1 and with the property that the opposite lattice L^o does not contain an infinite independent subset or, in other words, the lattice L does not contain an infinite coindependent subset. An element h of L is a uniform element of L^o if and only if h is proper (i.e., $h \neq 1$) and $a \vee b \neq 1$ for all proper elements $a, b \in 1/h$. Such an element h is called *hollow*.

THEOREM 2.2.37. *Let* $L \in \mathcal{M}_1$ *be a non-zero lattice which does not contain an infinite coindependent subset. Then there exists a finite coindependent set* H *of hollow elements of* L *such that* $\bigwedge H \in S(L)$*. Moreover,* $|H|$ *is an invariant of* L*.*

Proof. Apply Corollary 2.2.5 and Theorem 2.2.8(2) to the lattice L^o. $\square$

If $L \in \mathcal{M}_1$ is a non-zero lattice with no infinite coindependent subset then we shall call the invariant $|H|$ in Theorem 2.2.37 the *dual Goldie dimension* or *dual uniform dimension* of L and will denote it by $u^o(L)$. In case L is the zero lattice we define the dual Goldie dimension of L to be 0, and in case L is a lattice with an infinite coindependent subset we define the dual Goldie dimension of L by $u^o(L) = \infty$. Then Corollary 2.2.6 gives that every lattice with finite dual Goldie dimension is weakly supplemented. We shall leave to the reader to deduce from the above results for Goldie dimension corresponding results for dual Goldie dimension.

2.3. Krull dimension and Gabriel dimension

The idea of *dimension* is fundamental in many parts of Mathematics. Very intuitively, each kind of dimension *"takes the measure"* of the involved concepts from Mathematics in the form of *numerical invariants, cardinal invariants,* or *ordinal invariants.* Usually, it measures the deviation of a certain system from some ideal situation, or how likely or unlikely a certain object is to enjoy a certain property, or the progress in some inductive procedure. The most important dimensions encountering in Algebra, and in particular in Ring and Module Theory, are *Krull dimension, Goldie dimension, Gabriel dimension, (co)homological dimension,* and *Gelfand-Kirillov dimension.*

We have studied in Section 2.2 the concept of Goldie dimension of lattices. In this section we shall discuss the notions of Krull dimension and Gabriel dimension of arbitrary posets.

Krull dimension: a brief history

- **1923**: *E. Noether* explores the relationship between chains of prime ideals and dimensions of algebraic varieties.
- **1928**: *W. Krull* develops Noether's idea into a powerful tool for arbitrary commutative Noetherian rings. Later, writers gave the name (*classical*) *Krull dimension* to the supremum of lengths of finite chains of prime ideals in a ring.
- **1962**: *P. Gabriel* [**52**] introduces an ordinal valued dimension which he named "Krull dimension" for objects in an Abelian category using a transfinite sequence of localizing subcategories.
- **1967**: *R. Rentschler and P. Gabriel* introduce the *deviation* of an arbitrary poset, but only for finite ordinals.
- **1970**: *G. Krause* introduces the ordinal valued version of the Rentschler-Gabriel definition, but only for modules over an arbitrary unital ring.
- **1972**: *B. Lemonnier* [**59**] introduces the general ordinal valued notion of *deviation* of an arbitrary poset, called in the sequel *Krull dimension*.
- **1973**: *R. Gordon and J. C. Robson* [**54**] give the name *Gabriel dimension* to Gabriel's original definition after shifting the finite values by 1, and provide an incisive investigation of the Krull dimension of modules and rings.
- **1985**: *M. Pouzet and N. Zaguia* [**80**] introduce the more general concept of Γ-*deviation* of an arbitrary poset, where Γ is any set of ordinals.

The definition of the Krull dimension of a poset

The *Krull dimension* of a poset P (also called *deviation* of P and denoted by $\mathrm{dev}(P)$) is an ordinal number denoted by $k(P)$, which may or may not exist, and is defined recursively as follows:

- $k(P) = -1 \iff P = 0$, where -1 is assumed to be the predecessor of 0.
- $k(P) = 0 \iff P \neq 0$ and P is Artinian.
- Let $\alpha \geqslant 1$ be an ordinal number, and assume that we have already defined which posets have Krull dimension β for any ordinal $\beta < \alpha$. Then we define what it means for a poset P to have Krull dimension α: $k(P) = \alpha$ if and only if we have not defined $k(P) = \beta$ for some $\beta < \alpha$, and for any descending chain
$$x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant x_{n+1} \geqslant \ldots$$
of elements of P, $\exists\, n_0 \in \mathbb{N}$ such that $\forall\, n \geqslant n_0$, $k(x_n/x_{n+1}) < \alpha$, i.e., $k(x_n/x_{n+1})$ has previously been defined and it is an ordinal $< \alpha$.
- If no ordinal α exists such that $k(P) = \alpha$, we say that P does not have Krull dimension.

An alternative more compact equivalent definition of the Krull dimension of a poset is that involving the concept of an *Artinian poset relative* to a class of posets. If $\mathcal{X}$ is an arbitrary non-empty subclass of the class $\mathcal{P}$ of all posets, a poset P is said to be $\mathcal{X}$-*Artinian* if for every descending chain $x_1 \geqslant x_2 \geqslant \ldots$ in P, $\exists\, k \in \mathbb{N}$ such that $x_i/x_{i+1} \in \mathcal{X}$, $\forall\, i \geqslant k$. The notion of an $\mathcal{X}$-*Noetherian* poset is defined similarly.

For every ordinal $\alpha \geqslant 0$, we denote by $\mathcal{P}_\alpha$ the class of all posets having Krull dimension $< \alpha$. Then, it is easily seen that a poset P has Krull dimension an ordinal $\alpha \geqslant 0$ if and only if $P \notin \mathcal{P}_\alpha$ and P is $\mathcal{P}_\alpha$-Artinian. So, roughly speaking, the Krull dimension of a poset P measures how close P is to being Artinian.

The definition of the dual Krull dimension of a poset

The *dual Krull dimension* of a poset P (also called *codeviation* of P and denoted by $\mathrm{codev}(P)$), denoted by $k^o(P)$, is defined as being (if it exists!) the Krull dimension $k(P^o)$ of the opposite poset P^o of P. If α is an ordinal, then the notation $k(P) \leqslant \alpha$ (respectively, $k^o(P) \leqslant \alpha$) will be used to indicate that P has Krull dimension (respectively, dual Krull dimension) and it is less than or equal to α.

The existence of the dual Krull dimension $k^o(P)$ of a poset P is equivalent with the existence of the Krull dimension $k(P)$ of P in view of the following nice result of Lemonnier [59, Théorème 5, Corollaire 6]:

THEOREM 2.3.1. *An arbitrary poset P does not have Krull dimension if and only if P contains a copy of the (usually) ordered set $\mathbb{D} = \{\, m/2^n \mid m \in \mathbb{Z},\ n \in \mathbb{N} \,\}$ of dyadic real numbers. Consequently, P has Krull dimension if and only if P has dual Krull dimension.* □

Remember that
$$P \text{ is Artinian (respectively, Noetherian)} \iff k(P) \leqslant 0 \ \text{(respectively, } k^o(P) \leqslant 0).$$
So, we immediately deduce from Theorem 2.3.1 the following fact, which usually is proved in a more complicated way: *any Noetherian poset has Krull dimension.*

The following problem naturally arises:

Problem. *Let P be a poset with Krull dimension. Then P also has dual Krull dimension. How are the ordinals $k(P)$ and $k^o(P)$ related?* □

For other basic facts on the Krull dimension and dual Krull dimension of an arbitrary poset the reader is referred to [59] and [62].

Krull dimension and dual Krull dimension of modules and rings

Recall that for a module M one denotes by $\mathcal{L}(M)$ the lattice of all submodules of M. The following ordinals (if they exist) are defined in terms of the lattice $\mathcal{L}(M)$.

- *Krull dimension* of M: $k(M) := k(\mathcal{L}(M))$.
- *Dual Krull dimension* of M: $k^o(M) := k^o(\mathcal{L}(M))$.
- *Right Krull dimension* of R: $k(R) := k(R_R)$.
- *Right dual Krull dimension* of R: $k^o(R) := k^o(R_R)$.

The problem we presented above for arbitrary posets can be specialized to modules and rings as follows (see also the last subsection of Chapter 4):

Problem. *Compare the ordinals $k(M)$ and $k^o(M)$ of a given module M_R with Krull dimension. In particular, compare the ordinals $k(R)$ and $k^o(R)$ of a ring R with right Krull dimension.* □

The Faith's SI Theorem

A lovely 15-year-old result of Carl Faith [50] states:

Faith's SI Theorem (FT). *A module is Noetherian if and only if it is QFD and satisfies the* ACC *on its subdirectly irreducible submodules.* □

Recall that a module M_R is called *quotient finite dimensional* (or *QFD*) if any quotient module of M has finite Goldie dimension, i.e., the lattice $\mathcal{L}(M)$ of all its submodules is a QFD lattice. Also, recall that the subdirectly irreducible submodules of M have been called in Section 2.2 completely irreducible submodules, and their collection denoted by $\mathcal{I}^c(M)$. Thus, FT can be stated as follows:

$\mathcal{L}(M)$ *is a Noetherian poset* $\Longleftrightarrow$ M *is QFD and* $\mathcal{I}^c(M)$ *is a Noetherian poset.*

Since an arbitrary poset P is Noetherian if and only if it has dual Krull dimension $k^0(P) \leqslant 0$, FT can be reformulated in a dual Krull dimension setting as follows:

$$\mathrm{FT}_0: \quad k^0(\mathcal{L}(M)) \leqslant 0 \Longleftrightarrow \mathcal{L}(M) \text{ is a QFD lattice and } k^0(\mathcal{I}^c(M)) \leqslant 0.$$

The following natural problems related to FT arise:

(1) Investigate whether the dual FT^0 of the FT holds.

(2) Do the above reformulation FT_0 of the FT hold for an arbitrary ordinal α instead of 0, i.e., is the following statement

$$\mathrm{FT}_\alpha: \; k^0(\mathcal{L}(M)) \leqslant \alpha \Longleftrightarrow \mathcal{L}(M) \text{ is a QFD lattice and } k^0(\mathcal{I}^c(M)) \leqslant \alpha$$

true? A similar question for its dual FT^0_α.

(3) Extend (2) from the lattice $\mathcal{L}(M)$ to an arbitrary upper continuous modular lattice L.

(4) Apply (3) to Grothendieck categories and to module categories equipped with hereditary torsion theories.

We present below only a result concerning these four problems. More results on this topic may be found in [27] and [20].

Theorem 2.3.2. (The Latticial FT_n). *The following statements are equivalent for an upper continuous modular lattice L and a positive integer n.*

(1) $k^0(L) \leqslant n$.
(2) *L is both QFD and RSI, and* $k^0(\mathcal{J}^c(L)) \leqslant n$.

Proof. See [20, Theorem 1.18]. □

As we have seen in Corollary 2.2.34, the lattice $\mathcal{L}(M_R)$ is always RSI, so FT_n is true for $L = \mathcal{L}(M_R)$, M_R any module, and any $n \in \mathbb{N}$.

For a survey on the Faith's SI Theorem and its various extensions and dualizations the reader is referred to [12].

Classical Krull dimension of rings and posets

A crucial concept in *Commutative Algebra*, which actually originated the general concept of Krull dimension of an arbitrary poset, is that of *classical Krull dimension* cl.K.dim(R) of a commutative ring R, introduced by W. Krull in 1928; this is the supremum of length of chains of prime ideals of R, which is either a natural number or ∞. The following result relates the Krull dimension of a ring with its classical Krull dimension.

Theorem 2.3.3. *If R is a commutative ring with finite Krull dimension $k(R)$, then*

$$k(R) = \mathrm{cl.K.dim}(R).$$

Proof. See [54, Proposition 7.8]. □

A more accurate *ordinal valued* variant of the original Krull's classical dimension of a ring R, not necessarily commutative, is due to Gabriel (1962) [52] and Krause (1970). This can be easily carried out from the poset Spec(R) of all two-sided prime ideals of R to an arbitrary poset P to define a so called *classical Krull dimension* cl.K.dim(P) of P.

To do that, we denote by Max(P) the set, possibly empty, of all maximal elements of P. As in [1], we define recursively the following subsets of P:

$$P_0 := \mathrm{Max}(P),$$

and for any ordinal number $\alpha \geqslant 1$,

$$P_\alpha := \{\, x \in P \mid x < y \implies y \in \textstyle\bigcup_{\beta<\alpha} P_\beta \,\}.$$

Thus, we obtain an ascending chain

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \dots$$

of subsets of P, called the *Krull filtration* of P. There exists a least ordinal $\lambda(P)$ such that $P_{\lambda(P)} = P_{\lambda(P)+1}$. Clearly, $P_{\lambda(P)} \neq \varnothing \iff \mathrm{Max}(P) \neq \varnothing$. We say that P has *classical Krull dimension* if $P = P_{\lambda(P)}$, and in this case $\lambda(P)$ is called the *classical Krull dimension* of P and is denoted by cl.K.dim(P).

Lemma 2.3.4. *For any poset P and any ordinal α, either $P_\alpha = \varnothing$ or P_α is a Noetherian poset.*

Proof. Assume that $P_\alpha \neq \varnothing$. Let $\varnothing \neq X \subseteq P_\alpha$, and let β be the least element of the set $\{\,\gamma \leqslant \alpha \,|\, X \cap P_\gamma \neq \varnothing\,\}$ of ordinals. Clearly β is a non limit ordinal. Let $x \in X \cap P_\beta$. We claim that $x \in \operatorname{Max}(X)$. Indeed, if $\beta = 0$ then $x \in P_0 = \operatorname{Max}(P)$, so $x \in \operatorname{Max}(X)$. If $\beta > 0$ and $x < y$ with $y \in X$, then $y \in P_{\beta-1}$, so $y \in X \cap P_{\beta-1}$, which contradicts the definition of β. □

Proposition 2.3.5. *A poset P has classical Krull dimension if and only if P is Noetherian.*

Proof. If P has classical Krull dimension, then $P = P_{\lambda(P)}$, so P is Noetherian by Lemma 2.3.4.

Conversely, assume that P is a Noetherian poset and $P \neq P_{\lambda(P)}$. Then, there exists a maximal element x of $P \setminus P_{\lambda(P)}$. It follows that $x \in P_{\lambda(P)+1}$ because $x < y$ implies $y \in P_{\lambda(P)}$. Therefore $P = P_{\lambda(P)}$, i.e., P has classical Krull dimension. □

Finally, note that a different classical Krull dimension cl.k.dim(R) of a ring R, called the *little classical Krull dimension* of R is also defined in the literature. Its relations with $k(R)$ and cl.K.dim(R) are given in [**54**, Proposition 7.9]. Of course, this concept can be clearly extended to an arbitrary poset P to obtain the so called *little classical Krull dimension* cl.k.dim(P) of P.

Gabriel dimension

The notion of *Gabriel dimension* for objects of an Abelian category $\mathcal{A}$ has been introduced by P. Gabriel [**52**] under the name of *Krull dimension* using quotient categories and a transfinite sequence of localizing subcategories of $\mathcal{A}$. Roughly speaking, it measures the deviation of objects of $\mathcal{A}$ from the semi-Artinian type. R. Gordon and J. C. Robson [**54**] renamed the original Gabriel's name of this dimension into Gabriel dimension after shifting the finite values by 1.

We present below after [**4**], [**5**] the definition of the Gabriel dimension of an arbitrary poset P (for Gabriel dimension of upper continuous modular lattices, see also [**74**], [**81**]). Similarly to the Krull dimension of a poset, it is an ordinal number denoted by $g(P)$, which may or may not exist, and is defined recursively as follows:

- $g(P) = 0 \iff P$ is a trivial poset (this means that $a = b$ for all $a \leqslant b$ in P).
- Let $\alpha \geqslant 1$ be an ordinal number, and assume that we have already defined which posets do have Gabriel dimension β for any ordinal $\beta < \alpha$. Then we define what it means for a poset P to have *Gabriel dimension* $g(P) = \alpha$.
- First, we say that P is γ-*simple,* where γ is a non limit ordinal such that $1 \leqslant \gamma \leqslant \alpha$, if P is non trivial and for each $x \in P$ which is not minimal in P one has $g((x]) \not< \gamma$ (this means that we have not $g((x]) = \beta$ for some $\beta < \gamma$) and $g([x)) < \gamma$, i.e., $g([x))$ has been previously defined and it is an ordinal $< \gamma$.
- Now, we define $g(P) = \alpha$ if $g(P) \not< \alpha$ (i.e., we have not $g(P) = \beta$ for some $\beta < \alpha$) and for every $x < y$ in P, there exists $z \in P$ with $x < z \leqslant y$ such that z/x is γ-simple for some non limit ordinal $\gamma \leqslant \alpha$.
- If no ordinal α exists such that $g(P) = \alpha$, we say that P does not have Gabriel dimension.

REMARKS. From the definition above the following simple facts follow easily:

(1) $g(P) = \gamma$ for any γ-simple poset P.

(2) A poset P is 1-simple if and only if P is not trivial and each non minimal element of P is maximal. In case P has a least element 0 and a greatest element 1, then P is 1-simple if and only if $0 \neq 1$ and $P = \{0, 1\}$, i.e., P is a simple poset.

(3) A poset P has $g(P) = 1$ if and only if it is not trivial and for any $x < y$ in P there exists $z \in P$ with $x \prec z \leqslant y$. Thus, any non trivial Artinian poset has Gabriel dimension 1, in particular, $g(W) = 1$ for any well ordered set $W \neq 0$, and so $g(\mathbb{N}) = 1$ when $\mathbb{N}$ is ordered usually. The usually ordered set $\mathbb{Z}$ of rational integers is not Artinian, but nevertheless $g(\mathbb{Z}) = 1$.

(4) Each of the usually ordered posets $\mathbb{Q}$ of rational numbers, $\mathbb{R}$ of real numbers, and $\mathbb{D} = \{\, m/2^n \,|\, m \in \mathbb{Z},\ n \in \mathbb{N} \,\}$ of dyadic real numbers has no Gabriel dimension (see [4]).

(5) A non-zero upper continuous modular lattice L has $g(L) = 1$ if and only L is semi-Artinian. $\square$

The connections between the Krull dimension and the Gabriel dimension of a poset are given by the following two results.

PROPOSITION 2.3.6. *Any poset P having Krull dimension has also Gabriel dimension, and then*

$$g(P) \leqslant k(P) + 1.$$

PROOF. See [4, Proposition 2.1]. $\square$

THEOREM 2.3.7. *The following assertions are equivalent for an upper continuous modular lattice L.*

(1) *L has Gabriel dimension and L is* QFD.

(2) *L has Krull dimension.*

Moreover, in this case $g(L) = k(L)$ or $g(L) = k(L) + 1$.

PROOF. See [19, Theorem 5.2]. $\square$

CHAPTER 3

GROTHEDIECK CATEGORIES AND TORSION THEORIES

The main purpose of this chapter is two-fold: firstly, to present in a compact way some basic concepts of *Category Theory* as direct sum, direct product, subobject, quotient object, additive category, kernel, cokernel, image, coimage, Abelian category, quotient category, Grothendieck's axioms AB1, AB2, AB3, AB4, and AB5, leading to the definition of a *Grothendieck category,* and secondly, to discuss the concept of a *hereditary torsion theory,* needed in the last two chapters. We also discuss the renowned *Gabriel-Popescu Theorem.*

For more details and proofs, the reader is referred to [**52**], [**68**], and/or [**85**].

3.1. Categories and functors

In this section we present the concept of a category and illustrate it with quite many examples. Then, we define the notions of monomorphism, epimorphism, and isomorphism. The general concepts of direct product and direct sum in an arbitrary category are introduced just by taking as definitions the well-known universal properties of the direct product and direct sum of modules. Next, we give the definition of a covariant and contravariant functor, and present the concepts of a faithful, full, fully faithful, and representative functor. Finally, we discuss the concepts of a generator and cogenerator of a category.

Categories and subcategories

Definition. *A* category $\mathcal{C}$ *is given by*

(1) *A* class of objects, $\mathrm{Obj}\,(\mathcal{C})$.
(2) *A* class of morphisms: *for every ordered pair* (A, B) *of objects in* $\mathcal{C}$ *it is defined a* set (*possibly empty*) *denoted by* $\mathrm{Hom}_{\mathcal{C}}(A, B)$ *or* $\mathrm{Mor}_{\mathcal{C}}(A, B)$*, called the set of* morphisms *from* A *to* B*, such that*
$$\mathrm{Hom}_{\mathcal{C}}(A, B) \cap \mathrm{Hom}_{\mathcal{C}}(A', B') = \varnothing \textit{ for any } (A, B) \neq (A', B').$$
(3) *A* composition of morphisms, *i.e., for every triple* (A, B, C) *of objects in* $\mathcal{C}$ *there exists a mapping*
$$\mu_{A,B,C} : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C),$$
$$(f, g) \mapsto gf,$$
with the following properties:
   (i) *the composition is* associative*, i.e., for every* A, B, C, D *in* $\mathrm{Obj}\,(\mathcal{C})$ *and every* $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$*, one has*
$$h(gf) = (hg)f;$$

(ii) *there are* identities, *i.e., for any* $A \in \mathrm{Obj}\,(\mathcal{C})$, *there exists a morphism* $1_A \in \mathrm{Hom}_{\mathcal{C}}\,(A, A)$, *called the* identity *of* A, *with*

$$f1_A = 1_B f = f$$

for every $B \in \mathrm{Obj}\,(\mathcal{C})$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}\,(A, B)$. □

Note that the identity 1_A of A is uniquely determined. We often write $\mathrm{Hom}\,(A, B)$ for $\mathrm{Hom}_{\mathcal{C}}\,(A, B)$, and, more shortly, $A \in \mathcal{C}$ instead of $A \in \mathrm{Obj}\,(\mathcal{C})$.

For $f \in \mathrm{Hom}_{\mathcal{C}}\,(A, B)$ we write $f : A \longrightarrow B$ or $A \xrightarrow{f} B$, and we call A the *source* or *domain* of f and B the *target* or *codomain* of f.

Warning. In general, the objects of a category $\mathcal{C}$ are not always sets and the morphisms in $\mathcal{C}$ are not necessarily mappings; see Examples (12) and (13) below.

However, as for mappings. we shall use *diagrams* to illustrate morphisms. For example, the diagram below

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "l"'] & & B \arrow[d, "g"] \\
D \arrow[dr, "k"'] & & C \arrow[dl, "h"] \\
& E &
\end{tikzcd}$$

is called *commutative* if $h\,gf = k\,l$.

Definition. *A* subcategory *of a category* $\mathcal{C}$ *is a category* $\mathcal{D}$ *satisfying the following conditions.*

(1) $\mathrm{Obj}\,(\mathcal{D}) \subseteq \mathrm{Obj}\,(\mathcal{C})$.
(2) $\mathrm{Hom}_{\mathcal{D}}\,(A, B) \subseteq \mathrm{Hom}_{\mathcal{C}}\,(A, B)$ *for every* $A, B \in \mathrm{Obj}\,(\mathcal{D})$.
(3) *The composition of morphisms in* $\mathcal{D}$ *is the restriction of the composition in* $\mathcal{C}$.
(4) *For every* $A \in \mathrm{Obj}\,(\mathcal{D})$, *the identity of* A *in* $\mathcal{D}$ *is the same with the identity of* A *in* $\mathcal{C}$.

If additionally, $\mathrm{Hom}_{\mathcal{D}}\,(A, B) = \mathrm{Hom}_{\mathcal{C}}\,(A, B)$ *for every* $A, B \in \mathrm{Obj}\,(\mathcal{D})$, *then we say that* $\mathcal{D}$ *is a* full subcategory *of* $\mathcal{C}$, *and hence, a full subcategory is already determined by its objects.* □

Examples of categories

We are going to list some of the most common categories. First, note that in order to give a category $\mathcal{C}$ we have to specify the following four things:

$$\begin{cases} (i) & \text{the objects of } \mathcal{C}; \\ (ii) & \text{the sets } \mathrm{Hom}_{\mathcal{C}}\,(A, B) \text{ for every } A, B \in \mathcal{C}; \\ (iii) & \text{the composition of morphisms;} \\ (iv) & \text{the identities } 1_A \text{ for every } A \in \mathcal{C}. \end{cases}$$

(1) The category **Set** of *sets*:

- $\mathrm{Obj}\,(\mathbf{Set})$ is the class of all sets;

- $\mathrm{Hom}_{\mathbf{Set}}(A,B) := B^A$ is the set of all mappings from A to B; note that $\mathrm{Hom}_{\mathbf{Set}}(\varnothing, B)$ has only one element, the empty mapping, and $\mathrm{Hom}_{\mathbf{Set}}(A,\varnothing) = \varnothing$ for any non-empty set A;
- the composition in **Set** is the composition of mappings;
- 1_A is the identity mapping of A.

(2) The category $\mathbf{Set}_{Rel}$ of *sets with relations*:

- $\mathrm{Obj}\,(\mathbf{Set}_{Rel})$ is the class of all sets;
- $\mathrm{Hom}_{\mathbf{Set}_{Rel}}(A,B) := \mathcal{P}(A\times B)$ is the power set of $A\times B$, i.e., the set of all *relations* between A and B;
- the composition in $\mathbf{Set}_{Rel}$ is the composition of relations;
- 1_A is the relation Δ_A (the diagonal of A).

Clearly **Set** is a subcategory (but *not* full) of $\mathbf{Set}_{Rel}$.

(3) The *dual* category $\mathcal{C}^o$ of a category $\mathcal{C}$ is defined by

$$\mathrm{Obj}\,(\mathcal{C}^o) := \mathrm{Obj}\,(\mathcal{C})\,, \quad \mathrm{Hom}_{\mathcal{C}^o}(A,B) := \mathrm{Hom}_{\mathcal{C}}(B,A)$$

for any $A, B\in\mathcal{C}$, the composition and identities in $\mathcal{C}^o$ being clear.

An important problem is to describe concretely, modulo an equivalence of categories (see Section 3.3), the dual of a given category. For example, the dual of the category of all commutative rings with 1 is equivalent to the category of all *affine schemes* (by a result due to *Alexander Grothendieck*), and the dual of the category **Ab** of all Abelian groups is equivalent to the category of all *compact Abelian groups* (by the *Pontryagin Duality*).

(4) The category **Pos** of *posets,* mentioned in Section 1.1:

- $\mathrm{Obj}\,(\mathbf{Pos})$ is the class of all partially ordered sets (posets);
- $\mathrm{Hom}_{\mathbf{Pos}}(A,B)$ is the set of all order preserving mappings from A to B;
- the composition in **Pos** is the composition of mappings;
- 1_A is the identity mapping of A.

(5) The category **Lat** of *lattices,* mentioned in Section 1.1:

- $\mathrm{Obj}\,(\mathbf{Lat})$ is the class of all lattices;
- $\mathrm{Hom}_{\mathbf{Lat}}(A,B)$ is the set of all lattice morphisms from A to B;
- the composition in **Lat** is the composition of mappings;
- 1_A is the identity mapping of A.

(6) The categories **Top**, $\mathbf{Top}_0$, $\mathbf{Top}_1$, $\mathbf{Top}_2$, $\mathbf{Top}_3$, $\mathbf{Top}_4$:

- the objects are the *topological spaces* (respectively, topological spaces satisfying the separation axiom T_0, T_1, T_2, T_3, T_4, where T_0 stands for Kolmogoroff separation, T_1 for Fréchet separation, T_2 for Hausdorff separation, T_3 for regular separation, T_4 for normal separation);
- the morphisms from A to B are the continuous mappings from A to B;
- the composition is the composition of mappings;
- 1_A is the identity mapping of A.

(7) The categories **Gr** and **Ab** of all *groups* and *Abelian groups*, respectively. Clearly **Ab** is a full subcategory of **Gr**, and **Gr** is a subcategory of **Set**, but not full.

(8) The categories R-Mod and Mod-R of all *unital left R-modules* and *unital right R-modules,* respectively, where R is a unital ring.

(9) The categories R-**Alg** and R-**Lie** of all *associative unital R-algebras* and *Lie R-algebras*, respectively, where R is a commutative unital ring.

(10) The categories **Rin**, **Rinc**, **Rin1**, **Rinc1** of all *rings, commutative rings, unital rings, commutative unital rings,* respectively.

(11) The categories of all *affine schemes*, *differentiable manifolds*, *complex varieties*, *algebraic varieties*, *Banach spaces*, *Hilbert spaces*, *metric spaces*, etc.

Of course, the reader can easily guess what are the morphisms, compositions, and identities in each of the categories (7) - (11) listed above.

(12) Let $(M, \cdot)$ be an arbitrary *monoid.* We associate with $(M, \cdot)$ a category, denoted by $\widetilde{(M, \cdot)}$, as follows:

- $\mathrm{Obj}\,(\widetilde{(M, \cdot)})$ is a singleton set $\{*\}$;
- $\mathrm{Hom}_{\widetilde{(M,\cdot)}}(*, *) := M$, so the morphisms of this category are precisely the elements of M;
- the composition in $\widetilde{(M, \cdot)}$ is the given multiplication " $\cdot$ " in $(M, \cdot)$;
- the identity 1_* is the identity element of M.

(13) Let $(E, \leqslant)$ be a *quasi-ordered set.* This means that E is a non-empty set endowed with a *reflexive* and *transitive* binary relation " $\leqslant$ ". We shall associate with $(E, \leqslant)$ a category $\widetilde{(E, \leqslant)}$ in the following manner:

- $\mathrm{Obj}\,(\widetilde{(E, \leqslant)}) := E$, so the objects of this category are precisely the elements of E;
- $\mathrm{Hom}_{\widetilde{(E,\leqslant)}}(x,\, y) := \begin{cases} \varnothing & \text{if} \quad x \nleqslant y \\ \{(x,\, y)\} & \text{if} \quad x \leqslant y\,, \end{cases}$

  for every $x, y \in E$; so $\mathrm{Hom}\,(x, y)$ is $\varnothing$ if $x \nleqslant y$, and it has only one element, denoted by (x, y) if $x \leqslant y$;
- the composition of morphisms is defined by $(x, y)(y, z) = (x, z)$ whenever $x \leqslant y \leqslant z$ in E, and if $x \nleqslant y$ and z is arbitrary, or x is arbitrary and $y \nleqslant z$, then the composition is not defined;
- the identity of x is (x, x).

Note that the last two examples are small categories; a category $\mathcal{C}$ is called *small* if $\mathrm{Obj}\,(\mathcal{C})$ is a set. Moreover, the last two examples show that the morphisms of a category need not to be mappings.

Special morphisms in a category

Definitions. *Let $\mathcal{C}$ be a category, and let $f : A \longrightarrow B$ be a morphism in $\mathcal{C}$. The morphism f is called*

- monomorphism (*also denoted by* $A \overset{f}{\rightarrowtail} B$) *if for every object* $C \in \mathcal{C}$ *and morphisms* $g, h \in \mathrm{Hom}_{\mathcal{C}}(C, A)$, $fg = fh \implies g = h$ (*i.e.,* f *is* left cancellable)
$$C \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B;$$
- epimorphism (*also denoted by* $A \overset{f}{\twoheadrightarrow} B$) *if for every object* $D \in \mathcal{C}$ *and morphisms* $g, h \in \mathrm{Hom}_{\mathcal{C}}(B, D)$, $gf = hf \implies g = h$ (*i.e.,* f *is* right cancellable)
$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} D;$$
- bimorphism *if it is both monomorphism and epimorphism;*
- section (*or* coretraction) *if there exists* $B \xrightarrow{g} A$ *with* $gf = 1_A$
$$A \xrightarrow{f} B \xrightarrow{g} A;$$
- retraction *if there exists* $B \xrightarrow{h} A$ *with* $fh = 1_B$
$$B \xrightarrow{h} A \xrightarrow{f} B;$$
- isomorphism (*also denoted by* $A \overset{f}{\xrightarrow{\sim}} B$) *if it is both retraction and section, or equivalently if there exists* $B \xrightarrow{g} A$ *with* $gf = 1_A$ *and* $fg = 1_B$. $\square$

Observe that $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an epimorphism in $\mathcal{C}$ if and only if f, considered as an element in $\mathrm{Hom}_{\mathcal{C}^o}(B, A)$, is a monomorphism.

The notions in a category which are obtained one from another by reversing the arrows are called *dual*. For instance, the definition of an epimorphism is obtained by *dualizing* the notion of a monomorphism. The dual notion is often denoted with the prefix “co"; e.g., *coretraction* is dual to *retraction.* Note that the bimorphisms and the isomorphisms are dual to themselves; we say that they are *self dual.*

Examples. (1) In the full subcategory **Div** of the category **Ab**, consisting of all Abelian divisible groups, the canonical surjective mapping $\mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a monomorphism that is clearly not injective.

(2) In the category **Rinc1** of commutative rings with 1, the canonical injection mapping $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism that is clearly not surjective.

(3) Every morphism in the category $\widetilde{(E, \leqslant)}$ associated with a quasi-ordered set $(E, \leqslant)$ is a bimorphism, and, in case $(E, \leqslant)$ is a poset, then every isomorphism is an identity.

(4) The interested reader is invited to describe the monomorphisms and the epimorphisms in each of the categories: $\widetilde{(M, \cdot)}$, **Set**, Mod-R, **Gr**, **Top**, and $\mathbf{Top}_2$. $\square$

Subobjects and quotient objects

We are going to define the dual concepts of *subobject* and *quotient object* of an object X in an arbitrary category $\mathcal{C}$.

We say that two monomorphisms
$$X_1 \overset{f_1}{\rightarrowtail} X \quad \text{and} \quad X_2 \overset{f_2}{\rightarrowtail} X$$

in $\mathcal{C}$ are *equivalent* if there exists an isomorphism

$$X_1 \overset{\alpha}{\underset{\sim}{\longrightarrow}} X_2$$

making commutative the diagram

$$\begin{tikzcd} X_1 \arrow[dr, "f_1"] \arrow[d, "\alpha"'] & \\ X_2 \arrow[r, "f_2"'] & X \end{tikzcd}$$

Thus, we obtain an "equivalence" relation in the class of all monomorphisms having the target X, and an equivalence class of monomorphisms with target X is called *subobject* of X. In order to avoid some notational complications, we shall however make an abuse of notation (and language) by denoting a subobject by some representing monomorphism.

If $\alpha : Y \rightarrowtail X$ and $\beta : Z \rightarrowtail X$ are two subobjects of X, then we shall write $Y \leqslant Z$ (or $Y \subseteq Z$) if there exists a morphism $\gamma : Y \longrightarrow Z$ (necessarily unique, and monomorphism) such that the diagram

$$\begin{tikzcd} Y \arrow[dr, "\alpha"] \arrow[d, "\gamma"'] & \\ Z \arrow[r, "\beta"'] & X \end{tikzcd}$$

is commutative.

For any object X of $\mathcal{C}$ we denote by $\mathrm{Sub}(X)$ the class of all subobjects of X. Clearly, $\leqslant$ is a partial ordering in the class $\mathrm{Sub}(X)$, so we may say that $(\mathrm{Sub}(X), \leqslant)$ is a "*big poset*". A category $\mathcal{C}$ is called *locally small* if the class $\mathrm{Sub}(X)$ is a set for each $X \in \mathcal{C}$, and in this case, it is actually a *true* poset.

Dually one defines the notion of *quotient object* of an object in a category $\mathcal{C}$. We shall leave to the reader its details.

Direct product and direct sum in a category

The well-known universal properties of the direct product and direct sum of modules can be taken *mutatis mutandis* as definitions of the concepts of *direct product* and *direct sum* in an arbitrary category. More precisely, we have the following:

DEFINITION. *Let $\mathcal{C}$ be a category, and let $(A_i)_{i\in I}$ be a family of objects of $\mathcal{C}$. A* direct product *of this family is defined as being a pair $(P, (p_i)_{i\in I})$ consisting of an object P of $\mathcal{C}$ and a family of morphisms*

$$p_i : P \longrightarrow A_i, \quad i \in I,$$

such that, for every $X \in \mathcal{C}$ and every family $(f_i)_{i\in I}$ of morphisms $f_i : X \longrightarrow A_i$, $i \in I$,

$$\begin{tikzcd} P \arrow[r, "p_i"] & A_i \\ & X \arrow[u, "f_i"'] \arrow[ul, dotted, "f"] \end{tikzcd}$$

there exists a unique morphism $f : X \to P$ making commutative all the diagrams on the left side, i.e., $p_i f = f_i$, $\forall\, i \in I$. $\square$

Proposition 3.1.1. *Let $\mathcal{C}$ be a category, and let $(P,\, (p_i)_{i\in I})$ and $(P',\, (p'_i)_{i\in I})$ be two direct products of a given family $(A_i)_{i\in I}$ of objects of $\mathcal{C}$. Then, there exists a unique morphism $u : P \longrightarrow P'$ such that for every $i \in I$, the diagram*

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[dr, "p_i"'] & P' \arrow[d, "p'_i"] \\ & A_i \end{tikzcd}$$

is commutative. Moreover, u is an isomorphism.

Proof. For every $i \in I$ consider the following diagram.

$$\begin{tikzcd} P' \arrow[r, "p'_i"] & A_i \\ & P \arrow[u, "p_i"'] \arrow[ul, dotted, "u"] \end{tikzcd}$$

By definition, there exists a unique morphism u making this diagram commutative for each $i \in I$, since $(P',\, (p'_i)_{i\in I})$ is a direct product of $(A_i)_{i\in I}$.

Consider now the following diagram and take into account that $(P,\, (p_i)_{i\in I})$ is a direct product of $(A_i)_{i\in I}$.

$$\begin{tikzcd} P \arrow[r, "p_i"] & A_i \\ & P' \arrow[u, "p'_i"'] \arrow[ul, dotted, "v"] \end{tikzcd}$$

Then, by definition, there exists a unique morphism v making all these diagrams commutative.

Putting now together for each $i \in I$ the two commutative triangles above, we obtain the following bigger diagram which is also commutative.

$$\begin{tikzcd} P \arrow[r, "p_i"] \arrow[d, "u"] \arrow[dd, bend right, "1_P"'] & A_i \\ P' \arrow[ur, "p'_i"] \arrow[d, "v"] & \\ P \arrow[uur, "p_i"'] & \end{tikzcd}$$

But also 1_P makes the bigger diagram commutative, so using again the fact that $(P,\, (p_i)_{i\in I})$ is a direct product of $(A_i)_{i\in I}$, we deduce that

$$v\, u = 1_P.$$

In a similar manner we have $u\, v = 1_{P'}$, and consequently u is an isomorphism. $\square$

Using Proposition 3.1.1, we deduce that, if for a given family $(A_i)_{i\in I}$ of objects of a category $\mathcal{C}$ there exists at least a direct product, then it is uniquely determined up to an isomorphism. We can therefore, once for ever, choose one from all these direct products, denote it by $\prod_{i\in I} A_i$ (we renounce to indicate the p_i's, $p_i : \prod_{j\in I} A_j \longrightarrow A_i$, called the *canonical projections*, that appear in the definition of a direct product), and declare it as *the* (and not "a") *direct product* of the given family $(A_i)_{i\in I}$ of objects of

$\mathcal{C}$. For instance, in Mod-R the direct product of a family $(M_i)_{i\in I}$ of R-modules can be chosen as being precisely the usual Cartesian product $\prod_{i\in I} M_i$ of the family $(M_i)_{i\in I}$.

We leave to the reader the pleasure to formulate *mutatis mutandis* the definition of a *direct sum* of a family $(A_i)_{i\in I}$ of objects of $\mathcal{C}$, following the universal property of the direct sum of a family of modules. It is easily seen that the notion of a direct sum in a category is *dual* to the notion of a direct product; therefore, the direct sum of a family $(A_i)_{i\in I}$ of objects of $\mathcal{C}$ is sometimes called the *coproduct* of $(A_i)_{i\in I}$, it is uniquely determined up to an isomorphism, and is denoted by

$$\bigoplus_{i\in I} A_i \qquad \text{or} \qquad \coprod_{i\in I} A_i.$$

When all A_i, $i \in I$, are equal to a certain object A, we use the notations A^I for the direct product and $A^{(I)}$ for the direct sum of the family $(A_i)_{i\in I}$ of objects of $\mathcal{C}$.

Definition. *A category $\mathcal{C}$ is said to be a* category with direct products (*respectively, a* category with direct sums) *if every family of objects of $\mathcal{C}$ has a direct product* (*respectively, a direct sum*). □

Examples. (1) In the category $\widetilde{(E, \leqslant)}$ associated with a poset $(E, \leqslant)$, for a given family $(e_i)_{i\in I}$ of elements of E there exists a direct sum s of the family $(e_i)_{i\in I}$ if and only if there exists $\bigvee_{i\in I} e_i$ in $(E, \leqslant)$, and in this case, $s = \bigvee_{i\in I} e_i$, and a direct product p if and only if there exists $\bigwedge_{i\in I} e_i$ in $(E, \leqslant)$, and in this case, $p = \bigwedge_{i\in I} e_i$.

(2) In the full subcategory $\mathbf{Ab}_{fg}$ of $\mathbf{Ab}$ consisting of all finitely generated Abelian groups there exist neither infinite direct sums nor infinite direct products (this means that any family $(A_i)_{i\in I}$, I infinite set, of non-zero objects of $\mathbf{Ab}_{fg}$ has neither direct sum nor direct product in $\mathbf{Ab}_{fg}$).

(3) For any non-zero unital ring R, the full subcategory $\mathcal{N}(R)$ of Mod-R consisting of all Noetherian right R-modules has neither infinite direct sums nor infinite direct products.

(4) In the full subcategory $\mathbf{Set}_f$ of $\mathbf{Set}$ consisting of all finite sets there exist neither infinite direct sums nor infinite direct products.

(5) In the full subcategory $\mathbf{Fiel}$ of $\mathbf{Rinc1}$ consisting of all fields there exist neither direct sums nor direct products.

(6) The direct product in $\mathbf{Rinc1}$ of a family of objects is the usual Cartesian product of rings, and the direct sum in $\mathbf{Rinc1}$ of two commutative unital rings A_1, A_2 is their tensor product $A_1 \otimes_{\mathbb{Z}} A_2$. The interested reader is invited to investigate the existence of arbitrary direct sums in the category $\mathbf{Rinc1}$ (see also [16]). □

Covariant and contravariant functors

In this subsection we give the definitions of a *covariant* and *contravariant functor* and illustrate them with some examples.

Definitions. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* covariant functor

$$T : \mathcal{C} \longrightarrow \mathcal{D}$$

consists of a pair $T = (T_o, T_m)$ of assignments for

- *objects:* $T_o : \mathrm{Obj}\,(\mathcal{C}) \longrightarrow \mathrm{Obj}\,(\mathcal{D}),\ A \mapsto T_o(A)$,
- *morphisms:* $T_m : \mathrm{Mor}(\mathcal{C}) \longrightarrow \mathrm{Mor}(\mathcal{D})$,

$$(f : A \longrightarrow B) \ \mapsto\ T_m(f) : T_o(A) \longrightarrow T_o(B),$$

where $\mathrm{Mor}(\mathcal{C})$ *denotes the class of all morphisms of* $\mathcal{C}$, *with the following two properties:*

(i) $T_m(1_A) = 1_{T_o(A)}$ *for every* $A \in \mathrm{Obj}\,(\mathcal{C})$;
(ii) $T_m(gf) = T_m(g)T_m(f)$ *for every morphisms* f, g *in* $\mathcal{C}$ *for which* gf *is defined, i.e.,*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "gf"'] & B \arrow[d, "g"] & \Longrightarrow & T_o(A) \arrow[r, "T_m(f)"] \arrow[dr, "T_m(gf)"'] & T_o(B) \arrow[d, "T_m(g)"] \\
 & C & & & T_o(C)
\end{tikzcd}$$

A contravariant functor $S : \mathcal{C} \longrightarrow \mathcal{D}$ *consists of a pair* $S = (S_o,\, S_m)$ *of assignments for*

- *objects:* $S_o : \mathrm{Obj}\,(\mathcal{C}) \longrightarrow \mathrm{Obj}\,(\mathcal{D}),\ A \mapsto S_o(A)$,
- *morphisms:* $S_m : \mathrm{Mor}(\mathcal{C}) \longrightarrow \mathrm{Mor}(\mathcal{D})$,

$$(f : A \longrightarrow B) \ \mapsto\ S_m(f) : S_o(B) \longrightarrow S_o(A),$$

with the following two properties:

(i) $S_m(1_A) = 1_{S_o(A)}$ *for every* $A \in \mathrm{Obj}\,(\mathcal{C})$;
(ii) $S_m(gf) = S_m(f)S_m(g)$ *for every morphisms* f, g *in* $\mathcal{C}$ *for which* gf *is defined, i.e.,*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "gf"'] & B \arrow[d, "g"] & \Longrightarrow & S_o(A) & S_o(B) \arrow[l, "S_m(f)"'] \\
 & C & & & S_o(C) \arrow[u, "S_m(g)"'] \arrow[ul, "S_m(gf)"]
\end{tikzcd}$$

Usually, instead of T_o *and* T_m *we shall use the same letter* T, *so we shall write* $T(A)$ *and* $T(f)$ *for objects* A *as well as for morphisms* f. □

EXAMPLES. (1) The obvious transition from any category $\mathcal{C}$ to its dual $\mathcal{C}^o$ defines clearly a contravariant functor

$$D : \mathcal{C} \longrightarrow \mathcal{C}^o.$$

(2) The composition of two functors, which is defined in a very obvious manner, is again a functor, that is covariant if both are either covariant or contravariant, and is contravariant if one is covariant and the other one is contravariant. In particular, for any contravariant functor $S : \mathcal{C}^o \longrightarrow \mathcal{D}$, the composition of the functor D defined above with S yields a covariant functor $\mathcal{C} \longrightarrow \mathcal{D}$. Thus, the contravariant functors $\mathcal{C} \longrightarrow \mathcal{D}$ are precisely the covariant functors $\mathcal{C}^o \longrightarrow \mathcal{D}$.

(3) For any category $\mathcal{C}$ and any fixed object A in $\mathcal{C}$ two important functors arise:

(i) $\mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \longrightarrow \mathbf{Set},\ X \longmapsto \mathrm{Hom}_{\mathcal{C}}(A,X)$

$$(X \xrightarrow{f} Y) \longmapsto (\mathrm{Hom}_{\mathcal{C}}(A,X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(A,f)} \mathrm{Hom}_{\mathcal{C}}(A,Y)),$$

where $\mathrm{Hom}_{\mathcal{C}}(A,f)$ works as follows:

$$(A \xrightarrow{u} X) \longmapsto (A \xrightarrow{fu} Y).$$

This is a covariant functor which is usually denoted by h^A or H^A.

(ii) $\mathrm{Hom}_{\mathcal{C}}(-,A) : \mathcal{C} \longrightarrow \mathbf{Set}$, defined in a similar way. This is a contravariant functor, and usually it is denoted by h_A or H_A.

(4) If we associate with any group G the Abelian group $G^a := G/[G,G]$, where $[G,G]$ is the commutator subgroup of G, then we obtain a covariant functor

$$-^a : \mathbf{Gr} \longrightarrow \mathbf{Ab}.$$

(5) Let $\varphi : R \longrightarrow S$ be a unital ring morphism. Then φ defines two important functors.

(i) $\varphi_\star : \text{Mod-}S \longrightarrow \text{Mod-}R,\ M_S \mapsto M_R$,
where M_R is the underlying Abelian group of M_S endowed with the right R-module structure given by

$$x \cdot r = x\varphi(r)\,, \quad \forall\, r \in R,\ x \in M.$$

This functor is called the *functor of restriction of scalars via* φ.

(ii) $\varphi^\star : \text{Mod-}R \longrightarrow \text{Mod-}S,\ M_R \mapsto M \otimes_R S$,
where $M \otimes_R S$ is considered as a right S-module in a canonical way. This functor is called the *functor of extension of scalars via* φ.

(6) For any category $\mathcal{C}$, one defines the identity functor $1_{\mathcal{C}}$ of $\mathcal{C}$ as follows:

$$1_{\mathcal{C}}(A) = A \qquad \text{and} \qquad 1_{\mathcal{C}}(f) = f$$

for any $A \in \mathcal{C}$ and any $f \in \mathrm{Mor}(\mathcal{C})$.

More generally, if $\mathcal{D}$ is a subcategory of $\mathcal{C}$, then we have the obvious inclusion functor

$$\mathcal{D} \overset{i}{\hookrightarrow} \mathcal{C}.$$

(7) A category $\mathcal{C}$ is said to be *concrete* in case there is a “mapping"

$$\gamma : \mathcal{C} \longrightarrow \mathbf{Set}$$

such that

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \subseteq \mathrm{Hom}_{\mathbf{Set}}(\gamma(A),\gamma(B)), \quad \forall\, A,\, B \in \mathcal{C},$$

$$1_A = 1_{\gamma(A)}\,, \quad \forall\, A \in \mathcal{C},$$

and the composition of morphisms in $\mathcal{C}$ is the usual composition of mappings.

Thus, the usual categories $\mathbf{Ab}$, $\mathbf{Gr}$, Mod-R, $\mathbf{Rin}$, R-$\mathbf{Alg}$, $\mathbf{Top}$, etc., are all concrete categories. Note that γ is in fact a covariant functor, called the *forgetful functor.* $\square$

Special functors

A functor $T : \mathcal{C} \longrightarrow \mathcal{D}$ is said to *preserve* a property of an object $A \in \mathrm{Obj}\,(\mathcal{C})$ (respectively, of a morphism $f \in \mathrm{Mor}(\mathcal{C})$) if $T(A)$ (respectively, $T(f)$) again has the same property. The functor T is said to *reflect* a property of an object A (respectively, of a morphism f) if whenever $T(A)$ (respectively, $T(f)$) has this property, then this is also true for A (respectively, f).

For any covariant (respectively, contravariant) functor

$$T : \mathcal{C} \longrightarrow \mathcal{D}$$

and for any pair (A, B) of objects of $\mathcal{C}$ we have a (set) mapping

$$T_{A,\,B} : \mathrm{Hom}_{\mathcal{C}}\,(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}\,(T(A), T(B))$$

$$(\text{respectively, } T_{A,\,B} : \mathrm{Hom}_{\mathcal{C}}\,(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}\,(T(B), T(A)))$$

defined by

$$T_{A,\,B}(f) \;:=\; T(f).$$

This mapping will be used below.

Definitions. *A covariant or contravariant functor* $T : \mathcal{C} \longrightarrow \mathcal{D}$ *is called*

- faithful *if* $T_{A,\,B}$ *is injective for all* $A,\ B \in \mathcal{C}$;
- full *if* $T_{A,\,B}$ *is surjective for all* $A,\ B \in \mathcal{C}$;
- fully faithful *if* T *is full and faithful;*
- embedding *if the assignment* $T : \mathrm{Mor}(\mathcal{C}) \longrightarrow \mathrm{Mor}(\mathcal{D})$ *is injective;*
- representative *if for every* $D \in \mathcal{D}$, *there exists* $A \in \mathcal{C}$ *with* $T(A) \simeq D$. □

Generating objects

We present first a general definition of the concept of an $\mathcal{U}$-generated object that has meaning in an arbitrary category $\mathcal{C}$.

Definition 1. *Let* $\mathcal{C}$ *be an arbitrary category, and let* $\mathcal{U}$ *be a full subcategory of* $\mathcal{C}$. *An object* $A \in \mathcal{C}$ *is said to be* $\mathcal{U}$-generated *or* generated by $\mathcal{U}$, *if for any* $B \in \mathcal{C}$ *and any* $A \overset{f}{\underset{g}{\rightrightarrows}} B$, $f \neq g$, $\exists U \in \mathcal{U}$ *and* $U \xrightarrow{h} A$ *such that*

$$U \xrightarrow{h} A \overset{f}{\underset{g}{\rightrightarrows}} B, \qquad fh \neq gh.$$

In case $\mathcal{U} = \{U\}$ *is a singleton we use the terminology:* A *is* U-generated, *or* U generates A, *or* U is a generator for A.

By $\mathrm{Gen}\,(\mathcal{U})$ *we shall denote the class of all objects in* $\mathcal{C}$ *which are generated by* $\mathcal{U}$. *For* $\mathcal{U} = \{U\}$ *we write* $\mathrm{Gen}\,(U)$ *instead of* $\mathrm{Gen}\,(\mathcal{U})$.

If $\mathcal{C} = \mathrm{Gen}\,(\mathcal{U})$ *then we say that* $\mathcal{U}$ *is a* class of generators *for* $\mathcal{C}$, *and if* $\mathcal{C} = \mathrm{Gen}\,(U)$ *we call* U *a* generator *for* $\mathcal{C}$. □

Proposition 3.1.2. *Let* $\mathcal{C}$ *be an arbitrary category, and let* $U \in \mathcal{C}$.

(1) *The following assertions are equivalent for an object* $A \in \mathcal{C}$.
 (a) U *generates* A.

(b) *The mapping*

$$H^U_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathbf{Set}}(H^U(A), H^U(B))$$

$$f \mapsto H^U(f), \quad H^U(f)(h) = fh,\ h \in H^U(A),$$

is injective for every $B \in \mathcal{C}$, *where* H^U *is the covariant functor*

$$\mathrm{Hom}_{\mathcal{C}}(U, -) : \mathcal{C} \longrightarrow \mathbf{Set},\ X \longmapsto \mathrm{Hom}_{\mathcal{C}}(U, X).$$

(2) U *is a generator for* $\mathcal{C}$ *if and only if the covariant functor* H^U *is faithful.*

Proof. (1) The mapping $H^U_{A,B}$ is injective if and only if for every $f \neq g$ in $\mathrm{Hom}_{\mathcal{C}}(A, B)$, we have

$$H^U(f) \neq H^U(g),$$

i.e., the mappings

$$H^U(f) : H^U(A) \longrightarrow H^U(B) \text{ and } H^U(g) : H^U(A) \longrightarrow H^U(B)$$

are distinct; this means that $\exists\, h \in H^U(A)$ with

$$fh = H^U(f)(h) \neq H^U(g)(h) = gh,$$

which is exactly the definition of the fact that U generates A when $B \in \mathcal{C}$ is arbitrary.

(2) Follows immediately from (1). □

Of course, we can dualize the definitions and results above for generators to obtain the corresponding ones for cogenerators. For example, *an object* $C \in \mathcal{C}$ *is a* cogenerator *for* $\mathcal{C}$ *if and only if the contravariant functor* $H_U = \mathrm{Hom}_{\mathcal{C}}(-, U) : \mathcal{C} \longrightarrow \mathbf{Set}$ *is faithful.*

Not any category has a generator or cogenerator, e.g., the reader can easily check that **Fiel**, the full subcategory of **Rinc1** consisting of all fields, has no generator and no cogenerator.

The next definition is much closer to the definition involving direct sums of an U-generated module.

Definition 2. *Let* $\mathcal{C}$ *be any category with direct sums, and let* $\mathcal{U}$ *be a full subcategory of* $\mathcal{C}$. *An object* $A \in \mathcal{C}$ *is said to be* $\mathcal{U}$-generated *or* generated by $\mathcal{U}$ (*respectively,* finitely $\mathcal{U}$-generated *or* finitely generated by $\mathcal{U}$) *if there exists a set* Λ (*respectively, a finite set* Λ), *a family* $(U_\lambda)_{\lambda \in \Lambda}$ *of objects of* $\mathcal{U}$, *and an epimorphism*

$$\bigoplus\nolimits_{\lambda \in \Lambda} U_\lambda \longrightarrow\!\!\!\!\!\rightarrow A. \qquad \square$$

The next result shows that the apparently distinct Definitions 1 and 2 of $\mathcal{U}$-generated objects in a category $\mathcal{C}$ with direct sums are equivalent when $\mathcal{C}$ has a *zero object*. An object A of a category $\mathcal{C}$ is said to be an *initial* (respectively, *final*) *object* if for every $X \in \mathcal{C}$ there exists one and only one morphism from A to X (respectively, from X to A). A *zero object* of $\mathcal{C}$ is an object which is both initial and final.

We say that a subclass $\mathcal{U}_0$ of $\mathcal{U}$ is a *class of representatives* or a *representative class* of $\mathcal{U}$ if each $U \in \mathcal{U}$ is isomorphic to some member of $\mathcal{U}_0$.

Proposition 3.1.3. *Let* $\mathcal{C}$ *be a category with direct sums satisfying additionally the following condition:*

$(\star)$ $\qquad \mathrm{Hom}_{\mathcal{C}}(X, Y) \neq \varnothing$ *for every* $X, Y \in \mathcal{C}$.

(*Note that any category having a zero object satisfies* $(\star)$).

Let $\mathcal{U}$ be a full subcategory of $\mathcal{C}$ having a representative set $\mathcal{U}_0$. Then, the following statements are equivalent for an object $A \in \mathcal{C}$.

(1) *A is $\mathcal{U}$-generated in the sense of* Definition 1.
(2) *A is $\mathcal{U}_0$-generated in the sense of* Definition 1.
(3) *A is $\left(\bigoplus_{U\in\mathcal{U}_0} U\right)$-generated in the sense of* Definition 1.
(4) *A is $\mathcal{U}$-generated in the sense of* Definition 2.
(5) *A is $\mathcal{U}_0$-generated in the sense of* Definition 2.
(6) *A is $\left(\bigoplus_{U\in\mathcal{U}_0} U\right)$-generated in the sense of* Definition 2.

PROOF. See [8, Proposition 5.4]. □

3.2. Abelian categories

In this section we present in a compact way the basic concepts of additive category, kernel, cokernel, image, coimage, Abelian category, and Grothendieck's axioms AB1, AB2, AB3, AB4, AB5, leading to the concept of a Grothendieck category. For more details and proofs, the reader is referred to [85] and/or [68], [69].

Preadditive categories

DEFINITIONS. *A category $\mathcal{C}$ is said to be* preadditive *if the following conditions are satisfied.*

(1) *For every X, $Y \in \mathrm{Obj}\,(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}\,(X,Y)$ has a structure of Abelian group.*
(2) *For every X, Y, $Z \in \mathrm{Obj}\,(\mathcal{C})$, the composition mapping*

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}\,(X,Y) \times \mathrm{Hom}_{\mathcal{C}}\,(Y,Z) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}\,(X,Z) \\ (f,h) & \longmapsto & hf \end{array}$$

is bilinear, *i.e.,*

$$(h+k)f \;=\; (hf) \;+\; (kf)$$

and

$$h(f+g) \;=\; (hf) \;+\; (hg)$$

for every f, $g \in \mathrm{Hom}_{\mathcal{C}}\,(X,Y)$ and h, $k \in \mathrm{Hom}_{\mathcal{C}}\,(Y,Z)$, where "$+$" denote the algebraic operations in the corresponding groups.

A functor $T : \mathcal{C} \longrightarrow \mathcal{D}$ between two preadditive categories is said to be additive *if*

$$T(f+g) \;=\; T(f) \;+\; T(g)$$

for every X, $Y \in \mathrm{Obj}\,(\mathcal{C})$ and every f, $g \in \mathrm{Hom}_{\mathcal{C}}\,(X,Y)$. □

For any two objects X, Y of a preadditive category $\mathcal{C}$ we shall denote by $0_{X,Y}$, or more simple, by 0, the zero element of the Abelian group $\mathrm{Hom}_{\mathcal{C}}\,(X,Y)$.

EXAMPLE. If $(M,\cdot)$ is a monoid, then the category $\widetilde{(M,\cdot)}$ associated with this monoid is preadditive if and only if M is a ring. □

Note that in a preadditive category $\mathcal{C}$, an object X is a zero object if and only if $1_X = 0_{X,X}$.

Additive categories

Definition. *A category* $\mathcal{C}$ *is said to be* additive *if* $\mathcal{C}$ *is preadditive, has a zero object, and for every* $X, Y \in \operatorname{Obj}(\mathcal{C})$ *there exists* $X \bigoplus Y$. $\square$

It can be shown that in an additive category $\mathcal{C}$, for every finite family $(X_i)_{1\leqslant i\leqslant n}$ of objects, there exist their direct sum $\bigoplus_{1\leqslant i\leqslant n} X_i$ and direct product $\prod_{1\leqslant i\leqslant n} X_i$, and they are isomorphic.

Kernel, Cokernel, Image, Coimage

Definitions. *Let* $\mathcal{C}$ *be a preadditive category with a zero object, and let* $f : X \longrightarrow Y$ *be a morphism in* $\mathcal{C}$. *A* kernel *of* f *is a pair* (K, i) *consisting of an object* $K \in \mathcal{C}$ *and a morphism* $i : K \longrightarrow X$ *satisfying the following two conditions:*

$$\begin{tikzcd} K \arrow[r, "i"] & X \arrow[r, "f"] & Y \\ & A \arrow[u, "u"'] \arrow[ul, dotted, "\alpha"] & \end{tikzcd}$$

- $fi = 0$.
- *For any* $A \in \mathcal{C}$ *and any* $u \in \operatorname{Hom}_{\mathcal{C}}(A, X)$ *with* $fu = 0$, *there exists a unique morphism* $\alpha : A \longrightarrow K$ *making the above diagram commutative, i.e.,* $i\,\alpha = u$.

Dually, a cokernel *of* f *is a pair* (C, p) *consisting of an object* $C \in \mathcal{C}$ *and a morphism* $p : Y \longrightarrow C$ *satisfying the following two conditions:*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "p"] \arrow[d, "v"] & C \arrow[dl, dotted, "\beta"] \\ & B & \end{tikzcd}$$

- $pf = 0$.
- *For any* $B \in \mathcal{C}$ *and any* $v \in \operatorname{Hom}_{\mathcal{C}}(Y, B)$ *with* $vf = 0$, *there exists a unique morphism* $\beta : C \longrightarrow B$ *making the above diagram commutative, i.e.,* $\beta\, p = v$.

$\square$

It can be easily shown that if (K, i) is a kernel of a morphism $X \xrightarrow{f} Y$, then i is a monomorphism, and any two kernels of f represent the same subobject of X, i.e., they are equivalent. So, we can say *the* kernel of f when it exists, and we shall denote it just by $\operatorname{Ker}(f)$.

Similarly, if (C, p) is a cokernel of a morphism $X \xrightarrow{f} Y$, then p is an epimorphism, and any two cokernels of f represent the same quotient object of X, i.e., they are equivalent. So, we can say *the* cokernel of f when it exists, and we shall denote it just by $\operatorname{Coker}(f)$. If (X', i) is a subobject of an object X, then we denote $X/X' := \operatorname{Coker}(i)$.

Let $\mathcal{C}$ be a preadditive category having a zero object, and assume that each morphism in $\mathcal{C}$ has a kernel and a cokernel. For any morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ we introduce the following notation:

$$\operatorname{Im}(f) := \operatorname{Ker}(\operatorname{Coker}(f)) \quad \text{and} \quad \operatorname{Coim}(f) := \operatorname{Coker}(\operatorname{Ker}(f)),$$

which are called the *image* and *coimage* of f, respectively.

Consider now the following diagram:

$$\begin{tikzcd} A \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d, "q"'] & Y \arrow[r, "p"] & B \\ & C \arrow[ur, dotted, "v"] \arrow[r, dotted, "\overline{f}"'] & I \arrow[u, "j"'] & \end{tikzcd}$$

where

$$(A, i) = \mathrm{Ker}\,(f),\ (B, p) = \mathrm{Coker}\,(f),\ (I, j) = \mathrm{Im}\,(f),\ (C, q) = \mathrm{Coim}\,(f).$$

Since $(C, q) = \mathrm{Coker}\,(i)$, we have

$$f\,i\ =\ 0 \Longrightarrow \exists\, v : C \longrightarrow Y \text{ with } v\,q\ =\ f.$$

Now observe that $p\,v\,q = p\,f = 0 = 0\,q$, hence $p\,v = 0$ since q is an epimorphism. But $(I, j) = \mathrm{Ker}\,(p)$, hence $\exists\,\overline{f} : C \longrightarrow I$, called the *canonical morphism associated with* f, such that $j\,\overline{f}\ =\ v$, and so

$$f\ =\ j\,\overline{f}\,q.$$

Abelian categories

DEFINITIONS. *Let* $\mathcal{C}$ *be a category. We introduce the following axioms for* $\mathcal{C}$, *due to* Alexander Grothendieck.

AB1: $\mathcal{C}$ *is a preadditive category with zero object such that for every morphism* f *in* $\mathcal{C}$ *there exist* $\mathrm{Ker}\,(f)$ *and* $\mathrm{Coker}\,(f)$.

AB2: $\mathcal{C}$ *satisfies* AB1, *and for any morphism* $f : X \longrightarrow Y$ *in* $\mathcal{C}$, *the canonical morphism*

$$\overline{f} : \mathrm{Coim}\,(f) \longrightarrow \mathrm{Im}\,(f)$$

associated with f *is an isomorphism.*

AB3: $\mathcal{C}$ *is a category with arbitrary direct sums, i.e., for any set* I *and any family* $(X_i)_{i\in I}$ *of objects of* $\mathcal{C}$, *there exists* $\bigoplus_{i\in I} X_i$.

AB4: $\mathcal{C}$ *satisfies* AB3, *and for any set* I, *for any families* $(X_i)_{i\in I}$, $(Y_i)_{i\in I}$ *of objects of* $\mathcal{C}$, *and for any family* $(f_i)_{i\in I}$, $f_i : X_i \rightarrowtail Y_i$ *of monomorphisms, the canonical morphism*

$$\bigoplus_{i\in I} f_i : \bigoplus_{i\in I} X_i \longrightarrow \bigoplus_{i\in I} Y_i$$

is a monomorphism.

The category $\mathcal{C}$ *is said to be* preabelian *if it is additive and satisfies* AB1. *The category* $\mathcal{C}$ *is said to be* Abelian *if it is additive and satisfies* AB2. □

We shall denote by ABn^* $(n = 1, 2, 3, 4)$ the axiom ABn in the dual category $\mathcal{C}^o$ of $\mathcal{C}$. For instance, AB3* means exactly that $\mathcal{C}$ has direct products.

Note that if $\mathcal{C}$ is an Abelian category, then $\mathcal{C}^o$ is also Abelian. Indeed, AB1* = AB1 and AB2* = AB2.

EXAMPLES. (1) In the additive full subcategory mod-R of Mod-R consisting of all finitely generated R-modules, in general, kernels do not exist, and so, mod-R does not satisfy AB1.

(2) The category $\mathcal{A}$ of all Hausdorff topological Abelian groups is a preabelian category which is not an Abelian category. □

Sum and intersection of subobjects

Definitions. *Let $\mathcal{C}$ be an Abelian category, let $X \in \mathcal{C}$, and let $(X_i)_{i\in I}$ be a family of subobjects of X.*

(i) *If $\bigoplus_{i\in I} X_i$ exists, then we define and denote by $\sum_{i\in I} X_i$ the* sum *of the subobjects X_i as being the image of the canonical morphism α making commutative all the diagrams below, $i \in I$:*

$$\begin{tikzcd} X_i \arrow[r, "\varepsilon_i"] \arrow[dr, "\lambda_i"'] & \bigoplus_{j\in I} X_j \arrow[d, dotted, "\alpha"] \\ & X \end{tikzcd}$$

where the ε_i's are the canonical morphisms defining the direct sum $\bigoplus_{j\in I} X_j$ and the λ_i's are the monomorphisms defining the subobjects X_i, $i \in I$.

(ii) *If $\prod_{i\in I}(X/X_i)$ exists (where $X/X_i = \mathrm{Coker}\,(\lambda_i)$) then we define and denote by $\bigcap_{i\in I} X_i$ the* intersection *of the subobjects X_i as being the kernel of the canonical morphism β making commutative all the diagrams below, $i \in I$:*

$$\begin{tikzcd} \prod_{j\in I}(X/X_j) \arrow[r, "p_i"] & X/X_i \\ & X \arrow[u, "q_i"'] \arrow[ul, dotted, "\beta"] \end{tikzcd}$$

where the p_i's are the canonical morphisms defining the direct product $\prod_{j\in I}(X/X_j)$ and the q_i's are the epimorphisms defining the quotient objects X/X_i, $i \in I$. □

It is easy to see that if $(X_i)_{i\in I}$ is a family of subobjects of an object $X \in \mathcal{C}$ such that $\sum_{i\in I} X_i$ and $\bigcap_{i\in I} X_i$ exist, then $\sum_{i\in I} X_i$ is the least upper bound and $\bigcap_{i\in I} X_i$ is the greatest lower bound of the family $(X_i)_{i\in I}$ in the class $\mathrm{Sub}(X)$ of all subobjects of X (which is a set if $\mathcal{C}$ is a locally small category).

Note that if $\mathcal{C}$ is an arbitrary Abelian category, then, for any $X \in \mathcal{C}$, the "big poset" $\mathrm{Sub}(X)$ of all subobjects of X is actually a "*big lattice*", which is complete if the category $\mathcal{C}$ has direct sums and direct products. Even more, we have the following result:

Proposition 3.2.1. *Let $\mathcal{C}$ be an Abelian category, let $X \in \mathcal{C}$, and let $A, B, C \in \mathrm{Sub}(X)$ be such that $B \leqslant A$. Then*

$$A \cap (B \cup C) = B \cup (A \cap C),$$

i.e., $\mathrm{Sub}(X)$ is a "big" modular lattice with least element 0 and greatest element X.

Proof. See [85, Chapter IV, Proposition 5.3]. □

Proposition 3.2.2. *Any Abelian category having a generator is locally small.*

Proof. See [85, Chapter IV, Proposition 6.6]. □

Grothendieck categories

Definition. *A category* $\mathcal{C}$ *is said to satisfy the axiom* AB5 *if* $\mathcal{C}$ *is an Abelian category satisfying* AB3*, and moreover, for every object* $X \in \mathcal{C}$*, for every subobject* $Y \leqslant X$*, and for every family* $(X_i)_{i\in I}$ *of subobjects of* X *which is a* direct *family (this means, that* $\forall\, i,\, j \in I$*,* $\exists\, k \in I$ *such that* $X_i \leqslant X_k$ *and* $X_j \leqslant X_k$*) one has*

$$(*) \qquad \left(\sum_{i\in I} X_i\right) \cap Y = \sum_{i\in I} (X_i \cap Y).$$

A Grothendieck category *is any category having a generator and satisfying* AB5. □

Note that condition $(*)$ from the definition above means exactly that for any object X of a Grothendieck category $\mathcal{C}$, the class $\operatorname{Sub}(X)$ of all subobjects of X, which is a set by Proposition 3.2.2, is an upper continuous lattice. Observe that if $\mathcal{C}$ is an Abelian category with AB3 and a generator, then for every $X \in \mathcal{C}$, the set $\operatorname{Sub}(X)$ is a complete modular lattice.

From now on, we shall denote a Grothendieck category by $\mathcal{G}$, and for any object $X \in \mathcal{G}$ by $\mathcal{L}(X)$ the upper continuous modular lattice of all subobjects of X.

If $\mathbb{P}$ is any property on lattices, we say that an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Similarly, a subobject Y of an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the element Y of the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of a *Noetherian* object, *Artinian* object, *simple* object, *semi-Artinian* object, *uniform* object, *completely uniform* object, *compact* object, *subdirectly irreducible* object, *Goldie dimension* of an object, *Krull dimension* of an object, *Gabriel dimension* of an object, *pseudo-complement* subobject of an object, *essential* subobject of an object, *closed* subobject of an object, *complement* subobject of an object, *irreducible* subobject of an object, *completely irreducible* subobject of an object, etc.

Consequently, all the notions and results presented in Chapters 1 and 2 for an arbitrary lattice L can now be easily specialized for the particular case when $L = \mathcal{L}(X)$, where X is an object of a Grothendieck category $\mathcal{G}$. We leave this task to the reader.

We have seen in Proposition 2.1.9 that the compact elements of the lattice $\mathcal{L}(M)$ of all submodules of a right R-module M are precisely the finitely generated submodules of M. For this reason, the compact objects X of a Grothendieck category $\mathcal{G}$ (i.e., the objects X for which the lattice $\mathcal{L}(X)$ is compact) are called *finitely generated.* These are exactly those objects $X \in \mathcal{G}$ having the property that whenever $X = \sum_{i\in I} X_i$ for a family $(X_i)_{i\in I}$ of subobjects of X, there exists a finite subset J of I such that $X = \sum_{i\in J} X_i$. Similarly, for a complement subobject of an object $X \in \mathcal{G}$ one uses the well established term of a *direct summand* of X. Note that a Grothendieck category may have no non-zero finitely generated object, see below.

Definition. *We say that the Grothendieck category* $\mathcal{C}$ *is* locally finitely generated *if* $\mathcal{C}$ *has a set of generators consisting of finitely generated objects.* □

By [85, p. 122], a Grothendieck category $\mathcal{G}$ is locally finitely generated if and only if the lattices $\mathcal{L}(X)$ are compactly generated for all objects X of $\mathcal{G}$. There are plenty of such Grothendieck categories: for any $M \in \text{Mod-}R$, the full subcategory $\sigma[M]$ of all M-*subgenerated* modules of Mod-R is a locally finitely generated Grothendieck category. Recall that a module X_R is said to be M-subgenerated if X is isomorphic to a submodule of an M-generated module. In particular, $\sigma[R_R] = \text{Mod-}R$ is a locally finitely generated Grothendieck category with generator R_R. The reader is referred to [88] and/or [8] for more details on the category $\sigma[M]$.

A non-zero Grothendieck category which is locally finitely generated must possess simple objects, for such a category must possess a non-zero finitely generated object A, and a routine application of Zorn's Lemma shows that such an object A must have maximal proper subobjects (see Lemma 2.1.13(4) for the latticial counterpart of this categorical version of the renowned *Krull Lemma* from Module Theory). From this we infer the existence of simple objects. Notice that the converse of this property is, in general, not true, i.e., a Grothendieck category possessing simple objects is not necessarily locally finitely generated: an example of an indecomposable non locally finitely generated Grothendieck category possessing simple objects is constructed in [37]. By an *indecomposable category* we mean any category $\mathcal{C}$ which is not equivalent to a product of non-zero categories $\mathcal{C}_1 \times \mathcal{C}_2$.

Observe that not any Grothendieck category $\mathcal{G}$ is equivalent to a category $\sigma[N]$ for some ring A with identity and some $N \in$ Mod-A. Indeed, this happens whenever $\mathcal{G}$ contains no simple object, and an example of such a category is the following. Let R be an infinite direct product of copies of a field, and let $\mathcal{A}$ be the localizing subcategory of Mod-R consisting of all semi-Artinian R-modules. Then, the quotient category Mod-$R/\mathcal{A}$ has no simple object (see, e.g., [2, Remarks 1.4(1)]), in particular it has no non-zero finitely generated object. See the next section for the concepts of localizing subcategory, quotient category, and equivalence of categories.

PROBLEM. *Let $\mathcal{G}$ be a locally finitely generated Grothendieck category. Is it true that there exists a ring A with identity and $N \in$ Mod-A such that $\mathcal{G}$ be equivalent to the category $\sigma[N]$?* $\square$

3.3. Quotient categories

Clearly, for any ring R with identity element, the category Mod-R is a Grothendieck category. A procedure o construct new Grothendieck categories is to take the *quotient category* Mod-$R/\mathcal{T}$ of Mod-R modulo any of its localizing subcategories $\mathcal{T}$. The construction of the quotient category of Mod-$R/\mathcal{T}$, or more generally, of the quotient category $\mathcal{A}/\mathcal{S}$ of any locally small Abelian category $\mathcal{A}$ modulo any of its Serre subcategories $\mathcal{S}$ is quite complicated and goes back to Serre's "langage modulo $\mathcal{S}$" (1953), Grothendieck (1957), and Gabriel (1962) [52].

The aim of this section is to define the important concept of *quotient category* and to state the renowned *Gabriel-Popescu Theorem*. For more details and proofs, the reader is referred to [52] and [68]. For a different approach using the full subcategory of Mod-R consisting of all τ-closed R-modules, see [85, Chapter X].

Functorial morphisms

DEFINITION. *Let $\mathcal{C}$, $\mathcal{D}$ be two categories, and let $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{C} \longrightarrow \mathcal{D}$ be two covariant functors. By a* functorial morphism *or a* natural transformation

$$\varphi : F \longrightarrow G$$

from F *to* G *we mean a class of morphisms*

$$\varphi_X : F(X) \longrightarrow G(X), \ \ X \in \mathcal{C},$$

of $\mathcal{D}$ *such that for every morphism* $u : X \longrightarrow Y$ *in* $\mathcal{C}$*, the diagram*

$$F(X) @ > F(u) >> F(Y)$$

$$@V \varphi_X VV @VV \varphi_Y V$$

$$G(X) @ > G(u) >> G(Y)$$

is commutative.

If all φ_X, $X \in \mathcal{C}$, *are isomorphisms, then* $\varphi : F \longrightarrow G$ *is called a* functorial isomorphism *and we write* $\varphi : F \xrightarrow{\sim} G$. □

Clearly, if $\varphi : F \longrightarrow G$ and $\psi : G \longrightarrow H$ are two functorial morphisms, we can compose φ and ψ to obtain a functorial morphism

$$\psi \circ \varphi : F \longrightarrow H.$$

Also, for any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, the identities $1_{F(X)} : F(X) \longrightarrow F(X)$, $X \in \mathcal{C}$, yield the functorial isomorphism $\mathrm{Id} : F \longrightarrow F$.

Definition. *Two categories* $\mathcal{C}$ *and* $\mathcal{D}$ *are called* equivalent *if there exist covariant functors*

$$F : \mathcal{C} \longrightarrow \mathcal{D} \quad \text{and} \qquad G : \mathcal{D} \longrightarrow \mathcal{C}$$

with functorial isomorphisms

$$G \circ F \xrightarrow{\sim} 1_{\mathcal{C}} \quad \text{and} \qquad F \circ G \xrightarrow{\sim} 1_{\mathcal{D}}.$$

In this case, the functors F *and* G *are called* equivalences, *and* G *is called the* (equivalence) inverse *of* F. □

Proposition 3.3.1. *Let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be a covariant functor. Then* F *is an equivalence if and only if* F *is fully faithful and representative.*

Proof. See [85, Chapter IV, Proposition 1.1] or [88, 46.1(1)]. □

Exact functors

Definition. *A sequence of morphisms*

$$\ldots \to X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \to \ldots$$

in an Abelian category $\mathcal{C}$ *is said to be* exact at X_i *if* $\mathrm{Im}\,(f_{i-1}) = \mathrm{Ker}\,(f_i)$ *(equality as subobjects of* X_i*). The whole sequence is called* exact *if it is exact at each* X_i. □

Note that in Abelian categories one can work with exact sequences in essentially the same way as in module categories. Also note that the isomorphism theorems known in Mod-R hold in any Abelian category as well.

Definition. *Let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be an additive covariant functor between two Abelian categories* $\mathcal{C}$ *and* $\mathcal{D}$. *We say that* F *is* exact *if for any short exact sequence*

$$(*) \qquad 0 \to X' \xrightarrow{u} X \xrightarrow{v} X'' \to 0$$

in $\mathcal{C}$, *the sequence*

$$0 \to F(X') \xrightarrow{F(u)} F(X) \xrightarrow{F(v)} F(X'') \to 0$$

is an exact sequence in $\mathcal{D}$.

A contravariant functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ *is said to be* exact *if the associated covariant functor* $F^\circ : \mathcal{C}^o \longrightarrow \mathcal{D}$ *is* exact. □

Serre subcategories and localizing subcategories

We present below a bunch of definitions that will be used in this section for defining the concept of a quotient category, and in the next section for defining the concept of a torsion theory.

Definitions. *Let* $\mathcal{A}$ *be a full subcategory of* Mod-R. *We say that:*

(1) $\mathcal{A}$ *is* closed under subobjects *or* $\mathcal{A}$ *is a* hereditary class *if for any exact sequence*

$$0 \to A' \to A$$

in Mod-R *with* $A \in \mathcal{A}$, *it follows that* $A' \in \mathcal{A}$.

(2) $\mathcal{A}$ *is* closed under quotient objects *or* $\mathcal{A}$ *is a* cohereditary class *if for any exact sequence*

$$A \to A'' \to 0$$

in Mod-R *with* $A \in \mathcal{A}$, *it follows that* $A'' \in \mathcal{A}$.

(3) $\mathcal{A}$ *is* closed under extensions *if for any short exact sequence*

$$0 \to A' \to A \to A'' \to 0$$

in Mod-R *with* $A' \in \mathcal{A}$ *and* $A'' \in \mathcal{A}$, *it follows that* $A \in \mathcal{A}$.

(4) $\mathcal{A}$ *is a* Serre class *or* Serre subcategory *of* Mod-R *if it is hereditary, cohereditary, and closed under extensions.*

(5) $\mathcal{A}$ *is* closed under direct sums (*respectively,* closed under direct products) *if for any family of objects* $(A_i)_{i\in I}$, I *arbitrary set, with* $A_i \in \mathcal{A}$, $\forall\, i \in I$, *it follows that* $\bigoplus_{i\in I} A_i \in \mathcal{A}$ (*respectively,* $\prod_{i\in I} A_i \in \mathcal{A}$).

(6) $\mathcal{A}$ *is a* localizing subcategory *of* Mod-R *if it is a Serre class which is closed under direct sums.*

(7) $\mathcal{A}$ *is a* closed subcategory *of* Mod-R *if it is closed under subobjects, quotient objects, and direct sums.*

(8) $\mathcal{A}$ *is a* pretorsion class *if it is closed under quotient objects and direct sums.*

(9) $\mathcal{A}$ *is a* pretorsion-free class *if it is closed under subobjects and direct products.*

(10) $\mathcal{A}$ *is a* torsion class *if it is a pretorsion class which additionally is closed under extensions. So, a* hereditary torsion class *is exactly a localizing subcategory of* Mod-R.

(11) $\mathcal{A}$ *is a* torsion-free class *if it closed under subobjects, extensions and direct products.* □

Quotient categories

We are now going to present the concept of *quotient category.* We shall perform this construction starting with Mod-R, but it can be done "mutatis mutandis" for any locally small Abelian category instead of Mod-R.

Let $\mathcal{T}$ be an arbitrary *Serre subcategory* of Mod-R. We shall construct a new category called the *quotient category* of Mod-R *modulo* $\mathcal{T}$ and denoted by Mod-$R/\mathcal{T}$. This category is expected to have similar properties with that of a quotient module; so, Mod-$R/\mathcal{T}$ should be an Abelian category equipped with a covariant exact functor

$$T : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

such that T "kills" each $X \in \mathcal{T}$ (this means that $T(X) = 0, \forall X \in \mathcal{T}$), and moreover, T should be universal with these properties.

More precisely, we want to construct for the given Serre class $\mathcal{T}$ a pair (Mod-$R/\mathcal{T}$, T), where Mod-$R/\mathcal{T}$ is an Abelian category and

$$T : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

is a covariant exact functor, such that $T(X) = 0$, $\forall X \in \mathcal{T}$, and such that, for any Abelian category $\mathcal{A}$ and for any exact covariant functor

$$F : \text{Mod-}R \longrightarrow \mathcal{A}$$

with $F(X) = 0$, $\forall X \in \mathcal{T}$, there exists a unique functor H making commutative the diagram:

$$\begin{tikzcd} \text{Mod-}R \arrow[r, "T"] \arrow[dr, "F"'] & \text{Mod-}R/\mathcal{T} \arrow[d, dotted, "H"] \\ & \mathcal{A} \end{tikzcd}$$

The construction of the *quotient category* Mod-$R/\mathcal{T}$ of Mod-R *modulo* $\mathcal{T}$ is the following.

$$\text{Obj}\,(\text{Mod-}R/\mathcal{T}) := \text{Obj}\,(\text{Mod-}R).$$

For any M, $N \in$ Mod-R denote

$$I_{M,N} := \{\, (M', N') \mid M' \leqslant M,\ N' \leqslant N,\ M/M' \in \mathcal{T},\ N' \in \mathcal{T} \,\},$$

and define the following order relation in $I_{M,N}$:

$$(M', N') \preceq (M'', N'') \iff M'' \leqslant M' \text{ and } N' \leqslant N''.$$

Clearly, $I_{M,N}$ is an upward directed set.

Define now for M, $N \in$ Mod-R

$$\text{Hom}_{\text{Mod-}R/\mathcal{T}}\,(M, N) := \varinjlim_{(M',N') \in I_{M,N}} \text{Hom}_R\,(M', N/N').$$

THEOREM 3.3.2. *Let $\mathcal{T}$ be a Serre subcategory of* Mod-R *Then, the construction above defines an Abelian category* Mod-$R/\mathcal{T}$, *and the assignment*

$$\begin{array}{rccccc} T: & \text{Mod-}R & \longrightarrow & \text{Mod-}R/\mathcal{T} & & \\ & X & \longmapsto & T(X) = X & & \\ & (X \xrightarrow{f} Y) & \longmapsto & (T(f) : X \longrightarrow Y) & = & \textit{the image of } f \textit{ in} \\ & & & & & \textit{the inductive limit} \end{array}$$

is an exact functor. Moreover, the pair (Mod-$R/\mathcal{T}$, T) *has the above described universal property.*

Proof. See [52, Chapitre III] or [68, Corollario 25.10, Teorema 25.13]. □

The Gabriel-Popescu Theorem

Next, we are interested in knowing when the Abelian quotient category $\text{Mod-}R/\mathcal{T}$ of Mod-R modulo a Serre subcategory $\mathcal{T}$ is a Grothendieck category.

It can be shown (see, e.g., [52] or [68]) that the Serre subcategory $\mathcal{T}$ of Mod-R is a localizing subcategory of Mod-R if and only if the canonical functor

$$T : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

has a *right adjoint*

$$S : \text{Mod-}R/\mathcal{T} \longrightarrow \text{Mod-}R.$$

This means that for every $X \in \text{Mod-}R$ and $Y \in \text{Mod-}R/\mathcal{T}$ there exists a "functorial" isomorphism, i.e., *natural* in both first and second argument,

$$\text{Hom}_{\text{Mod-}R/\mathcal{T}}(TX, Y) \xrightarrow{\sim} \text{Hom}_R(X, SY),$$

and in this case $\text{Mod-}R/\mathcal{T}$ is a Grothendieck category.

Thus, we have a procedure to construct new Grothendieck categories starting with Mod-R; namely, by taking quotient categories of Mod-R modulo arbitrary localizing subcategories of Mod-R.

Roughly speaking, the renowned *Gabriel-Popescu Theorem*, discovered exactly fifty years ago, states that in this way we obtain *all* the Grothendieck categories. More precisely,

Theorem 3.3.3 (The Gabriel-Popescu Theorem). *Let $\mathcal{G}$ be an arbitrary Grothendieck category, and consider an arbitrary generator U of $\mathcal{G}$. Denote by R the ring $\text{End}_{\mathcal{G}}(U)$ of endomorphisms of U. Then there exists a localizing subcategory $\mathcal{T}$ of Mod-R such that*

$$\mathcal{G} \simeq \text{Mod-}R/\mathcal{T}.$$

Proof. See [68, pp. 130-138 and Osservazione 25.16] for an error-free and detailed proof. □

Notice that the ring R and the localizing subcategory $\mathcal{T}$ of Mod-R in Theorem 3.3.3 can be obtained in the following (non canonical) way. Let U be any generator of the Grothendieck category $\mathcal{G}$, and let R_U be the ring $\text{End}_{\mathcal{G}}(U)$ of endomorphisms of U. If

$$S_U : \mathcal{G} \longrightarrow \text{Mod-}R_U$$

is the functor $\text{Hom}_{\mathcal{G}}(U, -)$, then S_U has a left adjoint T_U, $T_U \circ S_U \simeq 1_{\mathcal{G}}$, and

$$\text{Ker}(T_U) := \{\, M \in \text{Mod-}R_U \mid T_U(M) = 0 \,\}$$

is a localizing subcategory of $\text{Mod-}R_U$. Take now as R any such R_U and as $\mathcal{T}$ such a $\text{Ker}(T_U)$.

3.4. Torsion theories

In this section we present some concepts and results on hereditary torsion theories that will be used in the sequel. The concept of *torsion theory* for Abelian categories has been introduced by S.E. Dickson [46] in 1966. For our purposes, we discuss it only for module categories in one of the many equivalent ways that can be done.

A *hereditary torsion theory* on Mod-R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of non-empty subclasses $\mathcal{T}$ and $\mathcal{F}$ of Mod-R such that $\mathcal{T}$ is a *localizing subcategory* of Mod-R and $\mathcal{F} = \{\, F_R \,|\, \mathrm{Hom}_R(T, F) = 0, \forall\, T \in \mathcal{T}\,\}$. Thus, any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is uniquely determined by its first component $\mathcal{T}$. Note that $\mathcal{F}$ is closed under subobjects, extensions, and direct products, i.e., is a torsion-free class.

The prototype of a hereditary torsion theory is the pair $(\mathcal{A}, \mathcal{B})$ in Mod-$\mathbb{Z}$, where $\mathcal{A}$ is the class of all torsion Abelian groups and $\mathcal{B}$ is the class of all torsion-free Abelian groups.

If I is a right ideal of a unital ring R, M is a right R-module, $r \in R$, and $x \in M$, then we denote

$$(I : r) =: \{\, a \in R \,|\, ra \in I\,\} \text{ and } \mathrm{Ann}_R(x) := \{\, a \in R \,|\, xa = 0\,\}.$$

A (right) *Gabriel filter* (or *Gabriel topology*) on R is a non-empty set F of right ideals of R satisfying the following two conditions:

- If $I \in F$ and $r \in R$, then $(I : r) \in F$;
- If I and J are right ideals of R such that $J \in F$ and $(I : r) \in F$ for all $r \in J$, then $I \in F$.

Each Gabriel filter F on R defines two classes of right R-modules

$$\mathcal{T}_F := \{\, M_R \,|\, \mathrm{Ann}_R(x) \in F, \forall\, x \in M\,\}$$

and

$$\mathcal{F}_F := \{\, M_R \,|\, \mathrm{Ann}_R(x) \notin F, \forall\, x \in M,\, x \neq 0\,\},$$

and the pair $(\mathcal{T}_F, \mathcal{F}_F)$ is a hereditary torsion theory on Mod-R. Conversely, to any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ we can associate the Gabriel filter

$$F_\tau := \{\, I \leqslant R_R \mid R/I \in \mathcal{T}\,\}.$$

It is well-known that the assignment $F \longmapsto (\mathcal{T}_F, \mathcal{F}_F)$ establishes a bijective correspondence between the set of all (right) Gabriel filters on R and the class of all hereditary torsion theories on Mod-R, with inverse correspondence given by $\tau \longmapsto F_\tau$ (see, e.g., [85, Chapter VI, Theorem 5.1]). In particular, the class of all hereditary torsion theories on Mod-R is actually a set.

Throughout this section $\tau = (\mathcal{T}, \mathcal{F})$ will be a fixed hereditary torsion theory on Mod-R. For any module M_R we denote

$$\tau(M) := \sum_{N \leqslant M,\, N \in \mathcal{T}} N.$$

Since $\mathcal{T}$ is a localizing subcategory of Mod-R, we have $\tau(M) \in \mathcal{T}$, and we call it the τ-*torsion submodule* of M. Note that, as for Abelian groups, we have

$$M \in \mathcal{T} \Longleftrightarrow \tau(M) = M \quad \text{and} \quad M \in \mathcal{F} \Longleftrightarrow \tau(M) = 0.$$

The members of $\mathcal{T}$ are called *τ-torsion modules*, while the members of $\mathcal{F}$ are called *τ-torsion-free modules*.

For further basic torsion-theoretic notions and results the reader is referred to [53] and/or [85].

The lattice $\mathrm{Sat}_\tau(M)$

For any M_R and any $N \leqslant M$ we denote

$$\mathrm{Sat}_\tau(M) := \{\, N \mid N \leqslant M,\ M/N \in \mathcal{F}\,\},$$

and call

$$\overline{N} := \bigcap \{\, C \mid N \leqslant C \leqslant M,\ M/C \in \mathcal{F}\,\}$$

the *τ-saturation* of N in M. We say that N is *τ-saturated* if $N = \overline{N}$. Note that $\overline{N}/N = \tau(M/N)$ and

$$\mathrm{Sat}_\tau(M) = \{\, N \mid N \leqslant M,\ N = \overline{N}\,\},$$

so $\mathrm{Sat}_\tau(M)$ is the set of all *τ-saturated* submodules of M, which explains the notation. Clearly, $\mathrm{Sat}_\tau(M)$ is a non-empty subset of the poset $\mathcal{L}(M)$ of all submodules of M ordered by inclusion $\subseteq$.

For any family $(N_i)_{i\in I}$ of elements of $\mathrm{Sat}_\tau(M)$ we set

$$\bigvee_{i\in I} N_i := \overline{\sum_{i\in I} N_i} \qquad \text{and} \qquad \bigwedge_{i\in I} N_i := \bigcap_{i\in I} N_i.$$

PROPOSITION 3.4.1. *For any module M_R the set $\mathrm{Sat}_\tau(M)$ of all τ-saturated submodules of M is an upper continuous modular lattice with respect to the inclusion $\subseteq$ and the operations $\bigvee$ and $\bigwedge$ defined above, and with least element $\tau(M)$ and greatest element M.*

PROOF. See [85, Chapter IX, Proposition 4.1] $\square$

Note that though $\mathrm{Sat}_\tau(M)$ is a subset of the lattice $\mathcal{L}(M)$ of all submodules of M, it is not a sublattice, because the sum of two τ-saturated submodules of M is not necessarily τ-saturated.

We present now three basic results on the lattice $\mathrm{Sat}_\tau(M)$ that will be very helpful in the next two chapters.

LEMMA 3.4.2. *Let M_R be a module, and let $N \in \mathcal{T}$ be a submodule of M. Then, the assignment $L \mapsto L/N$ provides a canonical lattice isomorphism*

$$\mathrm{Sat}_\tau(M) \xrightarrow{\sim} \mathrm{Sat}_\tau(M/N).$$

In particular, $\mathrm{Sat}_\tau(M) \xrightarrow{\sim} \mathrm{Sat}_\tau(M/\tau(M))$.

PROOF. For any $L \in \mathrm{Sat}_\tau(M)$ we have $N \leqslant \tau(M) \leqslant L$ and

$$(M/N)/(L/N) \simeq M/L \in \mathcal{F},$$

i.e., $L/N \in \mathrm{Sat}_\tau(M/N)$, and the result follows. $\square$

LEMMA 3.4.3. *Let M_R be a module, and let $X \in \mathrm{Sat}_\tau(M)$. Then, for any $N \leqslant M$ with $N \subseteq X$, the τ-saturation $\overline{N}$ of N in M coincides with the τ-saturation $\overline{N}_X$ of N in X.*

Proof. By definition,

$$\overline{N}/N = \tau(M/N) \quad \text{and} \quad \overline{N}_X/N = \tau(X/N).$$

Since $X/N \leqslant M/N$, we have $\overline{N}_X/N = \tau(X/N) \leqslant \tau(M/N) = \overline{N}/N$, so $\overline{N}_X \subseteq \overline{N}$.

In order to prove the opposite inclusion $\overline{N} \subseteq \overline{N}_X$, let $x \in \overline{N}$. Then, there exists a right ideal I of R such that $R/I \in \mathfrak{T}$ and $xI \subseteq N \subseteq X$. But $x \in M$ and $M/X \in \mathfrak{F}$, so $x \in X$. Because $xI \subseteq N$, we have $x + N \in \tau(X/N) = \overline{N}_X/N$, and then $x \in \overline{N}_X$, as desired. □

Lemma 3.4.4. *The following statements hold for a module M_R and submodules $P \subseteq N$ of M_R.*

(1) *The mapping*

$$\alpha : \mathrm{Sat}_\tau(N/P) \longrightarrow \mathrm{Sat}_\tau(\overline{N}/\overline{P}),\ X/P \mapsto \overline{X}/\overline{P},$$

is a lattice isomorphism.

(2) $\mathrm{Sat}_\tau(N) \simeq \mathrm{Sat}_\tau(\overline{N})$.

(3) *If $M/N \in \mathfrak{T}$, then $\mathrm{Sat}_\tau(M) \simeq \mathrm{Sat}_\tau(N)$.*

(4) *If $N, P \in \mathrm{Sat}_\tau(M)$, then the assignment $X \mapsto X/P$ defines a lattice isomorphism from the interval $[P, N]$ of the lattice $\mathrm{Sat}_\tau(M)$ onto the lattice $\mathrm{Sat}_\tau(N/P)$.*

Proof. (1) Let

$$\beta : \mathrm{Sat}_\tau(\overline{N}/\overline{P}) \longrightarrow \mathrm{Sat}_\tau(N/P),\ Y/\overline{P} \mapsto (Y \cap N)/P.$$

It can be easily checked that α and β are well defined mappings, that both are increasing, and that they are inverse to one another. So, α is an isomorphism of posets, and by Proposition 1.1.2, an isomorphism of lattices. Note that (1) is a specialization for the lattice $L = \mathrm{Sat}_\tau(M)$ of a more general result [**29**, Proposition 3.9].

(2) By (1) and Lemma 3.4.2 we have

$$\mathrm{Sat}_\tau(N) \simeq \mathrm{Sat}_\tau(N/0) \simeq \mathrm{Sat}_\tau(\overline{N}/\overline{0}) = \mathrm{Sat}_\tau(\overline{N}/\tau(M)) \simeq \mathrm{Sat}_\tau(\overline{N}).$$

(3) If $M/N \in \mathfrak{T}$, then $M/N = \tau(M/N) = \overline{N}/N$, so $\overline{N} = M$, and then, by (2), we have

$$\mathrm{Sat}_\tau(N) \simeq \mathrm{Sat}_\tau(\overline{N}) = \mathrm{Sat}_\tau(M).$$

(4) If $X \in [P, N]$ then $M/X \in \mathfrak{F}$, and so $(N/P)/(X/P) \simeq N/X \in \mathfrak{F}$, hence $X/P \in \mathrm{Sat}_\tau(N/P)$. Conversely, if $X/P \in \mathrm{Sat}_\tau(N/P)$, then $P \leqslant X \leqslant N$ and $(N/P)/(X/P) \simeq N/X \in \mathfrak{F}$. The exact sequence in Mod-R

$$0 \longrightarrow N/X \longrightarrow M/X \longrightarrow M/N \longrightarrow 0$$

with $N/X \in \mathfrak{F}$ and $M/N \in \mathfrak{F}$ yields $M/X \in \mathfrak{F}$, i.e., $X \in [P, N]$. Therefore, the lattices $\mathrm{Sat}_\tau(N/P)$ and $[P, N]$ are isomorphic as posets, and consequently also as lattices by Proposition 1.1.2. □

If $\mathbb{P}$ is any property on lattices, we say that a module M_R is/has τ-$\mathbb{P}$ if the lattice $\mathrm{Sat}_\tau(M)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of a *τ-Artinian* module, *τ-Noetherian* module, *τ-uniform* module, *τ-completely uniform* module, *τ-subdirectly irreducible* module, *τ-compact* module, *τ-compactly generated* module, *τ-Goldie dimension, τ-Krull dimension, τ-Gabriel dimension,* etc. Since the lattices $\mathrm{Sat}_\tau(M)$ and $\mathrm{Sat}_\tau(M/\tau(M))$ are canonically isomorphic by Lemma 3.4.2, we deduce that M_R is/has τ-$\mathbb{P}$ if and only if $M/\tau(M)$ is/has τ-$\mathbb{P}$.

We say that a submodule N of M_R is/has τ-$\mathbb{P}$ if its τ-saturation $\overline{N}$, which is an element of $\mathrm{Sat}_\tau(M)$, is/has $\mathbb{P}$. Thus, we obtain the concepts of a *τ-pseudo-complement* submodule of a module, *τ-complement* submodule of a module, *τ-essential* submodule of a module, *τ-closed* submodule of a module, *τ-irreducible* submodule of a module, *τ-completely irreducible* submodule of a module, etc. Since $\overline{N} = \overline{\overline{N}}$, it follows that N is/has τ-$\mathbb{P}$ if and only if $\overline{N}$ is/has τ-$\mathbb{P}$. In the sequel we shall use the well established term of a *τ-direct summand* of a module instead of that of a *τ-complement submodule* of a module.

Consequently, all the notions and results presented in Chapters 1 and 2 for an arbitrary lattice L can now be easily specialized for the particular case when $L = \mathrm{Sat}_\tau(M_R)$. We leave this task to the reader.

Example. Let R be a commutative ring with identity, and let S be a multiplicatively closed subset of R. Denote by $\mathcal{T}_S$ the full subcategory of Mod-R consisting of all modules M_R such that the module of fractions $S^{-1}M = 0$. Using the well-known properties of modules of fractions we deduce that $\mathcal{T}_S$ is actually a localizing subcategory of Mod-R. Denote by τ_S the (unique) hereditary torsion theory on Mod-R defined by $\mathcal{T}_S$. Then the τ_S-saturation of $N \leqslant M$ is precisely the S-saturation

$$\mathrm{Sat}_S(N) := \{\, x \in M \mid xs \in N \ \text{ for some } \ s \in S \,\}$$

of N, which explains the name. Consequently, $\mathrm{Sat}_{\tau_S}(M_R)$ consists exactly of all S-saturated submodules of M.

It is well-known that there exists an isomorphism of posets between the set of all S-saturated submodules of M and the lattice $\mathcal{L}(S^{-1}M_{S^{-1}R})$ of all submodules of the $S^{-1}R$-module $S^{-1}M$. So, the R-module M is τ_S-Noetherian (respectively, τ_S-Artinian) if and only if the $S^{-1}R$-module $S^{-1}M$ is Noetherian (respectively, Artinian). Moreover, there exists an equivalence of categories Mod-$R/\mathcal{T}_S \simeq$ Mod-$S^{-1}R$. $\square$

CHAPTER 4

THE HOPKINS-LEVITZKI THEOREM

The *Hopkins-Levitzki Theorem* (abbreviated H-LT) discovered independently in 1939, so 75 years ago, by C. Hopkins and J. Levitzki states that any right Artinian ring with identity is right Noetherian, or equivalently, it can be reformulated as:

Classical H-LT. *Let R be a right Artinian unital ring. Then any Artinian right R-module is Noetherian.*

In the last fifty years, especially in the 1970's, 1980's, and 1990's it has been generalized as follows:

Relative H-LT. *Let R be a ring with identity, and let τ be a hereditary torsion theory on* Mod-R. *If the ring R is τ-Artinian, then any τ-Artinian right R-module is τ-Noetherian.*

Absolute H-LT (or **Categorical H-LT**). *Let $\mathcal{G}$ be a Grothendieck category having an Artinian generator. Then any Artinian object of $\mathcal{G}$ is Noetherian.*

Latticial H-LT. *Let L be an arbitrary modular Artinian lattice with* 0. *Then L is Noetherian if and only if L satisfies two conditions, one of which guarantees that L has a good supply of essential elements and the second one ensures that there is a bound for the composition lengths of certain intervals of L.*

The aim of this chapter is to briefly explain all these aspects of the Classical Hopkins-Levitzki Theorem, their dual formulations, the connections between them, as well as to present other newer aspects of it involving the concepts of Krull and dual Krull dimension.

We shall also illustrate a *general strategy* which consists on putting a *module-theoretical* result into a *latticial frame* (we call it *latticization*), in order to translate that result to *Grothendieck categories* (we call it *absolutization*) and *module categories* equipped with *hereditary torsion theories* (we call it *relativization*).

4.1. The Classical Hopkins-Levitzki Theorem

The (Molien-)Wedderburn-Artin Theorem

One can say that the *Modern Ring Theory* begun in 1908, when *Joseph Henry Maclagan Wedderburn* (1882-1948) proved his celebrated Classification Theorem for finitely dimensional semisimple algebras over a field F (see [87]). Before that, in 1893, *Theodor Molien* or *Fedor Eduardovich Molin* (1861-1941) proved the theorem for $F = \mathbb{C}$ (see [65]).

In 1921, *Emmy Noether* (1882-1935) considers in her famous paper [75], for the first time in the literature, the *Ascending Chain Condition* (ACC)

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$$

for ideals in a commutative ring R.

In 1927, *Emil Artin* (1898-1962) introduces in [40] the *Descending Chain Condition* (DCC)

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$$

for left/right ideals of a ring and extends the Wedderburn Theorem to rings satisfying both the DCC and ACC for left/right ideals, observing that both ACC and DCC are a good substitute for finite dimensionality of algebras over a field:

THE (MOLIEN-)WEDDERBURN-ARTIN THEOREM. *A ring R is semisimple if and only if R is isomorphic to a finite direct product of full matrix rings over skew-fields*

$$R \simeq M_{n_1}(D_1) \times \ldots \times M_{n_k}(D_k). \qquad \square$$

Recall that by a *semisimple ring* one understands a ring which is left (or right) Artinian and has Jacobson radical or prime radical zero. Since 1927, the (Molien-) Wedderburn-Artin Theorem became a cornerstone of the Noncommutative Ring Theory.

In 1929, Emmy Noether observes (see [76, p. 643]) that the ACC in Artin's extension of the Wedderburn Theorem can be omitted. It took, however, ten years until it has been proved that always the DCC in a unital ring implies the ACC.

The Classical Hopkins-Levitzki Theorem (H-LT)

One of the most lovely results in Ring Theory is the *Hopkins-Levitzki Theorem*, abbreviated H-LT. This theorem, saying that any right Artinian ring with identity is right Noetherian, has been proved independently in 1939 by *Charles Hopkins* [57][1] (1902-1939) for left ideals and by *Jacob Levitzki* [61][2] (1904-1956) for right ideals. Almost surely, the fact that the DCC implies the ACC for one-sided ideals in a unital ring was unknown to both E. Noether and E. Artin when they wrote their pioneering papers on chain conditions in the 1920's.

An equivalent form of the H-LT, referred in the sequel also as the *Classical H-LT*, is the following one:

THE CLASSICAL H-LT. *Let R be a right Artinian ring with identity, and let M_R be a right module. Then M_R is an Artinian module if and only if M_R is a Noetherian module.*

[1]In fact, he proved that any left Artinian ring (called by him *MLI* ring) with left or right identity is left Noetherian (see [57, Theorems 6.4 and 6.7]).

[2]The result is however, surprisingly, neither stated nor proved in his paper, though in the literature, including our papers, the Hopkins' Theorem is also wrongly attributed to Levitzki. Actually, what Levitzki proved was that the ACC is superfluous in most of the main results of the original paper of Artin [40] assuming both the ACC and DCC for right ideals of a ring. This is also very clearly stated in the Introduction of his paper: "*In the present note it is shown that the maximum condition can be omitted without affecting the results achieved by Artin.*" Note that Levitzki considers rings which are not necessarily unital, so anyway it seems that he was even not aware about DCC implies ACC in unital rings; this implication does not hold in general in non unital rings, as the example of the ring with zero multiplication associated with any Prüfer quasi-cyclic p-group $\mathbb{Z}_{p^\infty}$ shows. Note also that though all sources in the literature, including Mathematical Reviews, indicate 1939 as the year of appearance of Levitzki's paper in *Compositio Mathematica*, the free reprint of the paper available at http://`www.numdam.org` indicates 1940 as the year when the paper has been published.

Proof. The standard proof of this theorem, as well as the original one of Hopkins [57, Theorem 6.4] for $M = R$, uses the Jacobson radical J of R. Since R is right Artinian, J is nilpotent and the quotient ring R/J is a semisimple ring. Let n be a positive integer such that $J^n = 0$, and consider the descending chain

$$M \supseteq MJ \supseteq MJ^2 \supseteq \ldots \supseteq MJ^{n-1} \supseteq MJ^n = 0$$

of submodules of M_R. Since the quotients MJ^k/MJ^{k+1} are all killed by J, each MJ^k/MJ^{k+1} becomes a right module over the semisimple ring R/J, and so, each MJ^k/MJ^{k+1} is a semisimple (R/J)-module.

Now, observe that M_R is Artinian (respectively, Noetherian) $\Longleftrightarrow$ all MJ^k/MJ^{k+1} are Artinian (respectively, Noetherian) R (or R/J)-modules. Since a semisimple module is Artinian if and only if it is Noetherian, it follows that M_R is Artinian if and only if it is Noetherian, which finishes the proof. □

Extensions of the H-LT

In the last fifty years, especially in the 1970's, 1980's, and 1990's the (Classical) H-LT has been generalized and dualized as follows:

- **1957**: *Fuchs* [51] shows that a left Artinian ring A, not necessarily unital, is Noetherian if and only if the additive group of A contains no subgroup isomorphic to the Prüfer quasi-cyclic p-group $\mathbb{Z}_{p^\infty}$.
- **1972**: *Shock* [83] provides necessary and sufficient conditions for a non unital Artinian ring and an Artinian module to be Noetherian; his proofs avoid the Jacobson radical of the ring and depend primarily upon the length of a composition series.
- **1976**: *Albu* and *Năstăsescu* [25] prove the *Relative H-LT*, i.e., the H-LT relative to a hereditary torsion theory, but only for commutative unital rings, and conjecture it for arbitrary unital rings.
- **1978-1979**: *Murase* [67] and *Tominaga* and *Murase* [84] show, among others, that a left Artinian ring A, not necessarily unital, is Noetherian if and only J/AJ is finite (where J is the Jacobson radical of R) if and only if the largest divisible torsion subgroup of the additive group of A is 0.
- **1979**: *Miller* and *Teply* [66] prove the *Relative H-LT* for arbitrary unital rings.
- **1979-1980**: *Năstăsescu* [70], [71] proves the *Absolute* or *Categorical H-LT*, i.e., the H-LT for an arbitrary Grothendieck category.
- **1980**: *Albu* [2] proves the *Absolute Dual H-LT* for commutative Grothendieck categories.
- **1982**: *Faith* [49] provides another module-theoretical proof of the Relative H-LT and also the Δ-Σ and *counter* versions of it.
- **1984**: *Albu* [3] establishes the *Latticial H-LT* for upper continuous modular lattices.
- **1996**: *Albu* and *Smith* [29] prove the *Latticial H-LT* for arbitrary modular lattices.
- **1996**: *Albu, Lenagan*, and *Smith* [23] establish a *Krull dimension-like* extension of the Classical H-LT and Absolute H-LT.
- **1997**: *Albu* and *Smith* [30] extend the result of Albu, Lenagan, and Smith [23] from Grothendieck categories to upper continuous modular lattices, using the technique of localization of modular lattices they developed in [29].

In the sequel we shall discuss in full detail all the extensions listed above of the HL-T for unital rings, as well as the connections between them.

4.2. The Relative and the Absolute Hopkins-Levitzki Theorem

In this section we discuss the relative and absolute counterparts of the Classical Hopkins-Levitzki Theorem and present a latticial strategy which will allow us in the next section to easily deduce in a unified manner both of them from a more general latticial approach.

The Relative Hopkins-Levitzki Theorem

The next result is due to Albu and Năstăsescu [25, Théorème 4.7] for commutative rings, conjectured for non commutative rings by Albu and Năstăsescu [25, Problème 4.8], and proved for arbitrary unital rings by Miller and Teply [66, Theorem 1.4].

THEOREM 4.2.1. (RELATIVE H-LT). *Let R be a ring with identity, and let τ be a hereditary torsion theory on* Mod-R. *If R is a right τ-Artinian ring, then every τ-Artinian right R-module is τ-Noetherian.* □

Let us mention that the module-theoretical proofs available in the literature of the Relative H-LT, namely the original one in 1979 due to Miller and Teply [66, Theorem 1.4] and another one in 1982 due to Faith [49, Theorem 7.1 and Corollary 7.2], are very long and complicated.

The importance of the Relative H-LT in investigating the structure of some relevant classes of modules, including injectives as well as projectives, is revealed in [26] and [49], where the main body of both these monographs deals with this topic.

Relativization

The Relative H-LT nicely illustrates a general direction in Module Theory, namely the so called *Relativization*. Roughly speaking, this topic deals with the following matter:

> *Given a property $\mathbb{P}$ in the lattice $\mathcal{L}(M_R)$ investigate the property $\mathbb{P}$ in the lattice* $\mathrm{Sat}_\tau(M_R)$.

Since more than forty years module theorists were dealing with the following problem:

> *Having a theorem $\mathbb{T}$ on modules, is its relativization τ-$\mathbb{T}$ true?*

As we mentioned just after the statement of the Relative H-LT, its known module-theoretical proofs are very long and complicated; so, the relativization of a result on modules is not always a simple job, and as this will become clear with the next statement, sometimes it may be even impossible.

METATHEOREM. *The relativization $\mathbb{T} \rightsquigarrow \tau$-$\mathbb{T}$ of a theorem $\mathbb{T}$ in Module Theory is not always true/possible.*

PROOF. Consider the following nice theorem of Lenagan [60, Theorem 3.2]:

$\mathbb{T}$: *If R has right Krull dimension then its prime radical* $\mathrm{N}(R)$ *is nilpotent.*

The relativization of $\mathbb{T}$ is the following:

τ-$\mathbb{T}$: *If R has right τ-Krull dimension then its τ-prime radical* $\mathrm{N}_\tau(R)$ *is τ-nilpotent.*

Recall that $\mathrm{N}_\tau(R)$ is the intersection of all τ-saturated two-sided prime ideals of R, and a right ideal I of R is said to be τ-nilpotent if $I^n \in \mathcal{T}$ for some integer $n > 0$.

The truth of the relativization τ-$\mathbb{T}$ of $\mathbb{T}$ has been asked by Albu and Smith [**28**, Problem 4.3]. Surprisingly, the answer is "no" in general, even if R is (left and right) Noetherian, by [**22**, Example 3.1]. This proves our Metatheorem.

However, τ-$\mathbb{T}$ is true for any ring R and any *ideal invariant* hereditary torsion theory τ, including any commutative ring R and any τ (see [**22**, Section 6]). $\square$

The Absolute Hopkins-Levitzki Theorem

The next result is due to Năstăsescu, who actually gave two different short nice proofs: [**70**, Corollaire 1.3] in 1979, based on the Loewy length, and [**71**, Corollaire 2] in 1980, based on the length of a composition series.

Theorem 4.2.2. (Absolute H-LT). *Let $\mathcal{G}$ be a Grothendieck category having an Artinian generator. Then any Artinian object of $\mathcal{G}$ is Noetherian.* $\square$

Recall that a *Grothendieck category* is an Abelian category $\mathcal{G}$ satisfying the axiom AB5 of Grothendieck and having a generator G (this means that for every object X of $\mathcal{G}$ there exist a set I and an epimorphism $G^{(I)} \twoheadrightarrow X$). A family $(U_j)_{j\in J}$ of objects of $\mathcal{G}$ is said to be a *family of generators* of $\mathcal{G}$ if $\bigoplus_{j\in J} U_j$ is a generator of $\mathcal{G}$ (see Proposition 3.1.3 for equivalent definitions of a family of generators in a category). The Grothendieck category $\mathcal{G}$ is called *locally Noetherian* (respectively, *locally Artinian*) if it has a family of Noetherian (respectively, Artinian) generators. Also, recall that an object $X \in \mathcal{G}$ is said to be *Noetherian* (respectively, *Artinian*) if the lattice $\mathcal{L}(X)$ of all subobjects of X is Noetherian (respectively, Artinian).

Note that J.E. Roos [**82**] has produced in 1969 an example of a locally Artinian Grothendieck category $\mathcal{C}$ which is not locally Noetherian; thus, the so called *Locally Absolute H-LT* fails. Even if a locally Artinian Grothendieck category $\mathcal{C}$ has a family of projective Artinian generators, then it is not necessarily locally Noetherian, as an example in [**63**] shows. However, the Locally Absolute H-LT is true if the family of Artinian generators of $\mathcal{C}$ is finite (because in this case $\mathcal{C}$ has an Artinian generator), as well as, by [**25**, Corollaire 4.38], if the Grothendieck category $\mathcal{C}$ is *commutative* (see the subsection on Absolute and Relative Dual H-LT in Section 4.4 for the definition of a commutative Grothendieck category).

Absolutization

The Absolute H-LT illustrates another general direction in Module Theory, namely the so called *Absolutization.* Roughly speaking, this topic deals with the following matter:

> *Given a property $\mathbb{P}$ on modules, investigate the property $\mathbb{P}$ on objects of a Grothendieck category.*

As for relativization, the following problem naturally arises:

> *Having a theorem $\mathbb{T}$ on modules, is its absolutization* abs-$\mathbb{T}$ *true?*

For example, the absolutization of the H-LT is true by Theorem 4.2.2, but, as we have seen at the end of Section 3.2, the absolutization of the property that any non-zero module has a simple factor module is not true.

We shall discuss now the interplay *Relativization* $\longleftrightarrow$ *Absolutization.* Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R. Then, because $\mathcal{T}$ is a localizing subcategory of Mod-R, one can form the quotient category Mod-$R/\mathcal{T}$, and denote by

$$T_\tau : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

the canonical functor from the category Mod-R to its quotient category Mod-$R/\mathcal{T}$.

Proposition 4.2.3. *With the notation above, for every module M_R there exists a lattice isomorphism*

$$\text{Sat}_\tau(M) \xrightarrow{\sim} \mathcal{L}(T_\tau(M)).$$

In particular, M is a τ-Noetherian (respectively, τ-Artinian) module if and only if $T_\tau(M)$ is a Noetherian (respectively, Artinian) object of Mod-$R/\mathcal{T}$.

Proof. See [**26**, Proposition 7.10]. □

We may also think that *Absolutization* is a technique to pass from τ-relative results in Mod-R to absolute properties in the quotient category Mod-$R/\mathcal{T}$ via the canonical functor T_τ : Mod-$R \longrightarrow$ Mod-$R/\mathcal{T}$. This technique is, in a certain sense, opposite to *Relativization*, meaning that absolute results in a Grothendieck category $\mathcal{G}$ can be translated, via the Gabriel-Popescu Theorem, into τ-relative results in Mod-R as follows.

Let U be any generator of the Grothendieck category $\mathcal{G}$, and let R_U be the ring $\text{End}_{\mathcal{G}}(U)$ of endomorphims of U. As we have mentioned just after Theorem 3.3.3, if $S_U : \mathcal{G} \longrightarrow \text{Mod-}R_U$ is the functor $\text{Hom}_{\mathcal{G}}(U, -)$, then S_U has a left adjoint T_U, $T_U \circ S_U \simeq 1_{\mathcal{G}}$, and $\text{Ker}(T_U) := \{\, M \in \text{Mod-}R_U \mid T_U(M) = 0 \,\}$ is a localizing subcategory of Mod-R_U. Let now τ_U be the hereditary torsion theory (uniquely) determined by the localizing subcategory $\text{Ker}(T_U)$ of Mod-R_U. Many properties of an object $X \in \mathcal{G}$ can now be translated into relative τ_U-properties of the right R_U-module $S_U(X)$; e.g., $X \in \mathcal{G}$ is an Artinian (respectively, Noetherian) object if and only if $S_U(X)$ is a τ_U-Artinian (respectively, τ_U-Noetherian) right R_U-module.

As mentioned before, the two module-theoretical proofs available in the literature of the Relative H-LT due to Miller and Teply [**66**] and Faith [**49**], are very long and complicated. On the contrary, the two categorical proofs of the Absolute H-LT due to Năstăsescu [**70**], [**71**] are very short and simple. We shall prove in Section 4.4 that *Relative H-LT* $\Longleftrightarrow$ *Absolute H-LT*; this means exactly that any of this theorems can be deduced from the other one. In this way we can obtain two short categorical proofs of the Relative H-LT.

However, some module theorists are not so comfortable with categorical proofs of module-theoretical theorems. Moreover, as we shall see in Section 5.3, statements like "*basically the same proof for modules works in the categorical setting*" may lead sometimes to wrong statements and results.

There exists an alternative for those people, namely the *latticial setting.* Indeed, if τ is a hereditary torsion theory on Mod-R and M_R is any module then $\text{Sat}_\tau(M)$ is an upper continuous modular lattice, and if $\mathcal{G}$ is a Grothendieck category and X is any object of $\mathcal{G}$ then $\mathcal{L}(X)$ is also an upper continuous modular lattice. Therefore, a strong reason to study such kinds of lattices exists.

A latticial strategy

Let $\mathbb{P}$ be a problem, involving subobjects or submodules, to be investigated in Grothendieck categories or in module categories with respect to hereditary torsion theories. Our *main strategy* in this direction since more than thirty years, we call *latticization,* consists of the following three steps:

I. *Translate/formulate,* if possible, the problem $\mathbb{P}$ into a *latticial setting.*
II. *Investigate* the obtained problem $\mathbb{P}$ in this *latticial frame.*
III. *Back to basics,* i.e., to Grothendieck categories and module categories equipped with hereditary torsion theories.

The advantage to deal in such a way is, in our opinion, that this is the most *natural* and *simple* approach as well, because we ignore the specific context of Grothendieck categories and module categories equipped with hereditary torsion theories, focussing only on those latticial properties which are relevant in our given specific categorical or relative module-theoretical problem $\mathbb{P}$. The best illustration of this approach is, as we shall see in Section 4.4, that both the *Relative H-LT* and the *Absolute H-LT* are immediate consequences of the so called *Latticial H-LT*, which will be amply discussed in the next section.

4.3. The Latticial Hopkins-Levitzki Theorem

The Classical/Relative/Absolute H-LT asks when a particular Artinian lattice $\mathcal{L}(M_R)/\mathrm{Sat}_\tau(M_R)/\mathcal{L}(X)$ is Noetherian. Our contention is that the natural setting for the H-LT and its various extensions is *Lattice Theory*, being concerned as it is with descending and ascending chains in certain lattices. Therefore we shall present in this section the Latticial H-LT which gives an exhaustive answer to the following more general question:

When an arbitrary Artinian modular lattice with 0 *is Noetherian?*

The condition $(\mathcal{E})$

We are interested in the following property that a lattice L may have ("$\mathcal{E}$" for Essential):

$(\mathcal{E})$ *For all* $a \leqslant b$ *in* L *there exists* $c \in L$ *such that* $b \wedge c = a$ *and* $b \vee c$ *is an essential element of* $[a)$.

Recall from Section 1.2 that a lattice $L \in \mathcal{M}_0$ is called *E-complemented* if for each $b \in L$ there exists $c \in L$ such that $b \wedge c = 0$ and $b \vee c$ is essential in L, and *completely E-complemented* if $[a)$ is E-complemented for all $a \in L$. So, condition $(\mathcal{E})$ means exactly that the lattice L is completely E-complemented. It is not clear whether the condition $(\mathcal{E})$ for a lattice L is equivalent or not with that of being E-complemented.

Lemma 4.3.1. *Any* QFD *lattice* $L \in \mathcal{M}_0$ *satisfies* $(\mathcal{E})$.

Proof. By Corollary 2.2.6, any lattice with finite Goldie dimension is E-complemented, so the result follows because the condition $(\mathcal{E})$ for L means exactly that the lattice $[a)$ is E-complemented for all $a \in L$. $\square$

Examples 4.3.2. (1) Noetherian lattices satisfy $(\mathcal{E})$ by Lemma 4.3.1 and Corollary 2.2.11(1).

(2) Any upper continuous modular lattice L satisfies $(\mathcal{E})$. Indeed, for any $a \in L$, the interval $1/a$ of L is upper continuous, so strongly pseudo-complemented, and then, also E-complemented. Thus, L is completely E-complemented, i.e., satisfies $(\mathcal{E})$.

(3) The set $\mathbb{N}$ of all natural numbers ordered by the usual divisibility is an Artinian modular lattice (even distributive), which does not satisfy $(\mathcal{E})$.

(4) Denote by K the lattice of all ideals of the subring R of $\mathbb{Q}$ consisting of all fractions a/b with $a, b \in \mathbb{Z}$, $b \neq 0$, $2 \nmid b$ and $3 \nmid b$. Then K^o is an Artinian lattice, it satisfies $(\mathcal{E})$ but is not pseudo-complemented. □

When an arbitrary Artinian modular lattice is Noetherian?

Lemma 4.3.3. *The following statements are equivalent for a complemented modular lattice L.*

(1) *L is Noetherian.*
(2) *L is Artinian.*
(3) *L has finite Goldie dimension.*
(4) *1 is a finite join of atoms of L.*
(5) *$1 = \bigvee A$ for some finite independent set A of atoms of L.*

Proof. (1) $\Longrightarrow$ (3) by Corollary 2.2.11(1).

(2) $\Longrightarrow$ (3) by Proposition 1.2.18(1) and Theorem 2.2.16.

(3) $\Longrightarrow$ (5) There exists a finite independent subset S of uniform elements of L such that $e := \bigvee S \in E(L)$. Since L is complemented it is clear that $e = 1$. Also, if u is a uniform element of L, then $u/0$ is a complemented lattice by Proposition 1.1.5, and hence u is an atom.

(5) $\Longrightarrow$ (4) is clear.

(4) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (2) by Corollary 2.1.3. □

Lemma 4.3.4. *Let $L \in \mathcal{M}_0$ be an E-complemented Artinian lattice. Then there exists an essential element e of L such that $e/0$ is a complemented Noetherian lattice.*

Proof. Clearly, we may assume that $L \neq \{0\}$. Let $e \in L$ be chosen minimal in the set $E(L)$ of all essential elements of L. Then $e \neq 0$. Let $0 < a \leqslant e$. There exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b \in E(L)$. Since $e \wedge (a \vee b) \in E(L)$, by the choice of e we have
$$e = e \wedge (a \vee b) = a \vee (e \wedge b).$$
Thus $e/0$ is an Artinian complemented lattice. By Lemma 4.3.3, $e/0$ is also Noetherian, and we are done. □

In order to characterize Artinian lattices which are Noetherian, we introduce the following condition ("$\mathcal{BL}$" for Bounded Length) for lattices with a least element:

> $(\mathcal{BL})$ *There exists $n \in \mathbb{N}$ such that for all $x < y$ in L with $y/0$ having a composition series there exists $c_{xy} \in L$ such that $c_{xy} \leqslant y$, $c_{xy} \nleqslant x$, and $\ell(c_{xy}/0) \leqslant n$.*

The next result is the *Hopkins-Levitzki Theorem* for an arbitrary modular lattice with least element, that will be used in the next section to provide very short proofs of the *absolute* (or *categorical*) and *relative* counterparts of the *Classical Hopkins-Levitzki Theorem.*

Theorem 4.3.5. (Latticial H-LT). *Let $L \in \mathcal{M}_0$ be an Artinian lattice. Then L is Noetherian if and only if L satisfies both conditions* $(\mathcal{E})$ *and* $(\mathcal{BL})$.

Proof. If L is Noetherian, then take $c_{xy} = y$ for all $x < y$ in L. Observe that $\ell(y/0) \leqslant \ell(L)$, hence L satisfies $(\mathcal{BL})$, and by Examples 4.3.2(1), it also satisfies $(\mathcal{E})$.

Conversely, suppose that L satisfies both $(\mathcal{E})$ and $(\mathcal{BL})$. We can obviously suppose that $L \neq \{0\}$, for otherwise it is nothing to prove. By Lemma 4.3.4, the condition $(\mathcal{E})$ ensures the existence of an ascending chain

$$0 = e_0 < e_1 \leqslant e_2 \leqslant \ldots$$

of elements of L such that e_i/e_{i-1} is Noetherian and $e_i \in E([e_{i-1}))$ for all $i \geqslant 1$. Suppose that n is the positive integer in the $(\mathcal{BL})$ condition for L. If $L \neq (e_n]$ then $e_n < e_{n+1}$. According to the condition $(\mathcal{BL})$, there exists $c \in L$ such that

$$c \leqslant e_{n+1},\ c \nleqslant e_n, \text{ and } \ell(c/0) \leqslant n.$$

Now

$$0 = c \wedge e_0 \leqslant c \wedge e_1 \leqslant \ldots \leqslant c \wedge e_{n-1} \leqslant c \wedge e_n < c.$$

Suppose that $c \wedge e_i = c \wedge e_{i-1}$ for some $1 \leqslant i \leqslant n$. Then

$$(c \vee e_{i-1}) \wedge e_i = (c \wedge e_i) \vee e_{i-1} = (c \wedge e_{i-1}) \vee e_{i-1} = e_{i-1}$$

implies that $c \vee e_{i-1} = e_{i-1}$ since $e_i \in E([e_{i-1}))$, and then $c \leqslant e_{i-1} \leqslant e_n$, which is a contradiction. Thus

$$0 = c \wedge e_0 < c \wedge e_1 \ \ldots < c \wedge e_{n-1} < c \wedge e_n < c,$$

so that $\ell(c/0) \geqslant n+1$, which is again a contradiction. It follows that $L = (e_n]$, and hence L is Noetherian, as desired. □

Observe that if e is an element as in Lemma 4.3.4 and $L \neq \{0\}$, then there exists a positive integer n and a finite independent set $\{a_1, \ldots, a_n\}$ of atoms of L such that $e = \bigvee_{1 \leqslant i \leqslant n} a_i$. Further, for each atom a of L we have $a \wedge e \neq 0$, hence $a \leqslant e$, and consequently e is exactly the *socle* $\mathrm{Soc}(L)$ of L, i.e., the join of all atoms of L. Note that the given E-complemented Artinian lattice L need not be complete. Thus, the chain

$$0 = e_0 \leqslant e_1 \leqslant e_2 \leqslant \ldots$$

defined in the proof of Theorem 4.3.5 is nothing other than the so called *Loewy series* of L which is defined similarly as for modules, but, in our special setting, without using the upper continuity of L, as it is usually done.

Lattice generation

We now exhibit a natural situation, involving the concept of "lattice generation", where the condition $(\mathcal{BL})$ occurs. In order to define it, first recall some definitions and facts on module generation. If R is a unital ring and M, U are two unital right R-modules, then M is said to be *U-generated* if there exist a set I and an epimorphism

$U^{(I)} \twoheadrightarrow M$. The fact that M is U-generated can be equivalently expressed as follows: for every proper submodule N of M there exists a submodule P of M which is not contained in N, such that P is isomorphic to a quotient of the module U. Further, M is said to be *strongly U-generated* if every submodule of M is U-generated. These concepts can be naturally extended to arbitrary lattices as follows:

DEFINITIONS. *We say that a lattice $L \in \mathcal{L}_1$ is* generated *by a lattice $G \in \mathcal{L}_1$ (or is G-*generated*) if for every $a \neq 1$ in L there exist $c \in L$ and $g \in G$ such that $c \nleqslant a$ and $(c] \simeq 1/g$. A lattice $L \in \mathcal{L}$ is called* strongly generated *by a lattice $G \in \mathcal{L}_1$ (or* strongly G-generated*) if for every $b \in L$, the interval $(b]$ is G-generated, i.e., for all $a < b$ in L, there exist $c \in L$ and $g \in G$ such that $c \leqslant b$, $c \nleqslant a$, and $(c] \simeq 1/g$.* □

Recall from Section 1.1 that we have denoted by $\mathcal{L}$ (respectively, $\mathcal{L}_0$, $\mathcal{L}_1$) the class of all lattices (respectively, lattices with least element 0, lattices with greatest element 1). Every lattice $G \in \mathcal{L}_1$ is G-generated (take $c = 1$ and $g = 0$ in the definition above), and the zero lattice 0 is also G-generated for any $G \in \mathcal{L}_1$. Of course, as in [7], the above definitions can be obviously further extended from lattices to posets.

Clearly, if the R-module M is (strongly) U-generated, then the lattice $\mathcal{L}(M_R)$ is (strongly) $\mathcal{L}(U_R)$-generated, but not conversely.

LEMMA 4.3.6. *Let $L \in \mathcal{M}_1$ be an Artinian lattice. Then, the set*

$$N = \{\, a \,|\, 1/a \text{ is Noetherian} \,\}$$

has a (unique) least element a^*.

PROOF. Clearly $1 \in N$. Let a^* be a minimal element of N, and let b be an arbitrary element of N. Then $1/(a^* \wedge b)$ is Noetherian by Corollary 2.1.3(1), i.e., $a^* \wedge b \in N$, and so $a^* \wedge b = a^*$ by the minimality of a^*. Consequently $a^* \leqslant b$, which proves that a^* is the (unique) least element of N. □

For any Artinian lattice $L \in \mathcal{M}_1$ we shall denote by $\ell^*(L) := \ell(1/a^*)$, with a^* as in Lemma 4.3.6, the so called *reduced length* of L, Observe that a^* is the least element of L such that $1/a^*$ is a lattice of finite length.

PROPOSITION 4.3.7. *Let $L \in \mathcal{M}_0$ be such that L is strongly generated by an Artinian lattice $G \in \mathcal{M}_1$. Then L satisfies the condition $(\mathcal{BL})$.*

PROOF. Let $n = \ell^*(G)$, and let $x < y$ in L be such that $y/0$ has a composition series. There exist elements $z \in L$, $g \in G$ such that $z \leqslant y$, $z \nleqslant x$, and $z/0 \simeq 1/g$, so $1/g$ is Noetherian. By Lemma 4.3.6, $a^* \leqslant g$, and then $1/g \subseteq 1/a^*$. This implies that $\ell(z/0) = \ell(1/g) \leqslant \ell(1/a^*) = \ell^*(G) = n$. It follows that L satisfies $(\mathcal{BL})$. □

Combining Theorem 4.3.5 and Proposition 4.3.7 we have at once:

THEOREM 4.3.8. *Let $L \in \mathcal{M}_0$ be an Artinian lattice which is strongly generated by an Artinian lattice $G \in \mathcal{M}_1$. Then L is Noetherian if and only if L satisfies the condition $(\mathcal{E})$. In particular, if additionally L is upper continuous, then L is Noetherian.* □

When an arbitrary Noetherian modular lattice is Artinian?

The results of the previous subsection can be easily dualized by asking when a Noetherian lattice is Artinian. From now on, L will denote a modular lattice with a greatest element, i.e., $L \in \mathcal{M}_1$.

The dual properties of $(\mathcal{E})$ and $(\mathcal{BL})$ for a lattice $L \in \mathcal{M}_1$ are the following:

$(\mathcal{E}^o)$ *For all* $a \leqslant b$ *in* L *there exists* $c \in L$ *such that* $a \vee c = b$ *and* $a \wedge c$ *is a superfluous element of* $(\,b\,]$.

and

$(\mathcal{BL}^o)$ *There exists* $n \in \mathbb{N}$ *such that for all* $x < y$ *in* L *with* $1/x$ *having a composition series there exists* c_{xy} *in* L *such that* $x \leqslant c_{xy}$, $y \nleqslant c_{xy}$, *and* $\ell(1/c_{xy}) \leqslant n$.

Examples of modular lattices that satisfy or not the condition $(\mathcal{E}^o)$ can be easily obtained by taking the opposites of the lattices discussed in Examples 4.3.2; e.g., the opposite lattice $\mathbb{N}^o$ of the lattice $\mathbb{N}$ of all natural numbers ordered by the usual divisibility is Noetherian and does not satisfy $(\mathcal{E}^o)$.

Theorem 4.3.9. (Latticial Dual H-LT). *Let* $L \in \mathcal{M}_1$ *be a Noetherian lattice. Then* L *is Artinian if and only if* L *satisfies both conditions* $(\mathcal{E}^o)$ *and* $(\mathcal{BL}^o)$.

Proof. Observe that the opposite of a modular lattice (respectively, lattice of finite length) is also a modular lattice (respectively, lattice of finite length). Also

$$L \in \mathcal{M}_1 \text{ satisfies } (\mathcal{E}^o) \iff L^o \in \mathcal{M}_0 \text{ satisfies } (\mathcal{E}),$$

and

$$L \in \mathcal{M}_1 \text{ satisfies } (\mathcal{BL}^o) \iff L^o \in \mathcal{M}_0 \text{ satisfies } (\mathcal{BL}).$$

Now, the result follows immediately from Theorem 4.3.5. □

Lattice cogeneration

Recall that if R is a unital ring and M, U are two unital right R-modules, then M is said to be U-*cogenerated* if there exist a set I and a monomorphism $M \rightarrowtail U^I$. The fact that M is U-cogenerated can be equivalently expressed as follows: for any non-zero submodule N of M there exist a submodule P of M and a submodule U' of U such that $N \nsubseteq P$ and $M/P \simeq U'$. Further, we say that a module M is *strongly* U-*cogenerated* in case any quotient module of M is U-cogenerated. These concepts can be naturally extended to arbitrary lattices as follows:

Definitions. *A lattice* $L \in \mathcal{L}_0$ *is said to be* cogenerated *by a lattice* $C \in \mathcal{L}_0$ *or* C-cogenerated *if for any* $x \neq 0$ *in* L *there exist* $z \in L$ *and* $c \in C$ *with* $x \nleqslant z$ *and* $[z) \simeq c/0$. *A lattice* $L \in \mathcal{L}$ *is called* strongly cogenerated *by a lattice* $C \in \mathcal{L}_0$ *or* strongly C-cogenerated *if for any* $y \in L$, *the interval* $[y)$ *is* C-*cogenerated, that is, for any* $y < x$ *in* L *there exist* $z \in L$ *and* $c \in C$ *such that* $y \leqslant z$, $x \nleqslant z$, *and* $[z) \simeq c/0$. □

Of course, the above definitions can be obviously further extended from lattices to posets. Observe that C-cogeneration is dual to G-generation:

$$L \text{ is } C\text{-cogenerated} \iff L^o \text{ is } C^o\text{-generated.}$$

The dual statements of Proposition 4.3.7 and Theorem 4.3.8 are the following:

PROPOSITION 4.3.10. *Let* $L \in \mathcal{M}_1$ *be such that* L *is strongly cogenerated by a Noetherian lattice* $G \in \mathcal{M}_0$. *Then* L *satisfies the condition* $(\mathcal{BL}^o)$. $\square$

THEOREM 4.3.11. *Let* $L \in \mathcal{M}_1$ *be a Noetherian lattice which is strongly cogenerated by a Noetherian lattice* $G \in \mathcal{M}_0$. *Then* L *is Artinian if and only if* L *satisfies the condition* $(\mathcal{E}^o)$. $\square$

Theorems 4.3.5 and 4.3.9 have the following consequence:

COROLLARY 4.3.12. *The following statements are equivalent for a lattice* $L \in \mathcal{M}_{0,1}$.

(1) L *has a composition series.*
(2) L *is Artinian, satisfies* $(\mathcal{BL})$, *and* $1/a$ *has finite Goldie dimension for all* $a \in L$.
(3) L *is Artinian, satisfies* $(\mathcal{BL})$, *and is upper continuous.*
(4) L *is Artinian and satisfies both* $(\mathcal{BL})$ *and* $(\mathcal{E})$.
(5) L *is Noetherian, satisfies* $(\mathcal{BL}^o)$, *and* $a/0$ *has finite dual Goldie dimension for all* $a \in L$.
(6) L *is Noetherian, satisfies* $(\mathcal{BL}^o)$, *and is lower continuous.*
(7) L *is Noetherian and satisfies both* $(\mathcal{BL}^o)$ *and* $(\mathcal{E}^o)$.

PROOF. Clearly $(1) \Longrightarrow (2)$. We have also $(1) \Longrightarrow (3)$ according to Corollary 2.1.15, $(2) \Longrightarrow (4)$ by Lemma 4.3.1, $(3) \Longrightarrow (4)$ by Examples 4.3.2(2), $(4) \Longrightarrow (1)$ by Theorem 4.3.5, and $(1) \Longleftrightarrow (7)$ by Theorem 4.3.9. The other equivalences follow by considering the dual lattice L^o of L. $\square$

4.4. Other aspects of the Hopkins-Levitzki Theorem

The aim of this section is two-fold: firstly, to discuss all the connections between the Classical, Relative, Absolute, and Latticial H-LT, and secondly, to present other aspects of the H-LT including the *Faith's* Δ-Σ and *counter* versions of the Relative H-LT, the *Dual H-LT,* as well as a *Krull dimension-like H-LT.* In particular, we show in a unified manner that both the Relative H-LT and Absolute H-LT are immediate consequences of the Latticial H-LT.

Latticial H-LT $\Longrightarrow$ Relative H-LT

As mentioned above, the module-theoretical proofs available in the literature of the Relative H-LT are very long and complicated. We present below a very short proof of it based on the Latticial H-LT.

So, let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R. Assume that R is τ-Artinian, and let M_R be a τ-Artinian module. The Relative H-LT states that M_R is a τ-Noetherian module.

Set $G := \mathrm{Sat}_\tau(R_R)$ and $L := \mathrm{Sat}_\tau(M_R)$. Then G and L are Artinian upper continuous modular lattices. We have to prove that M_R is a τ-Noetherian module, i.e., L is a Noetherian lattice. By Theorem 4.3.8, it is sufficient to check that L is strongly G-generated, i.e., for every $a < b$ in L, there exist $c \in L$ and $g \in G$ such that $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/g$.

Since $\mathrm{Sat}_\tau(M) \simeq \mathrm{Sat}_\tau(M/\tau(M))$ by Lemma 3.4.2, we may assume, without loss of generality, that $M \in \mathcal{F}$. Let $a = A < B = b$ in $L = \mathrm{Sat}_\tau(M_R)$. Then, there exists $x \in B \setminus A$. Set $C := \overline{xR}$ and $I := \mathrm{Ann}_R(x)$. We have $R/I \simeq xR \leqslant M \in \mathcal{F}$, so $R/I \in \mathcal{F}$, i.e., $I \in \mathrm{Sat}_\tau(R_R) = G$. By Lemma 3.4.4, we deduce that

$$[\,I,\,R\,] \simeq \mathrm{Sat}_\tau(R/I\,) \simeq \mathrm{Sat}_\tau(xR\,) \simeq \mathrm{Sat}_\tau(\overline{xR}\,) = \mathrm{Sat}_\tau(C) = [\,0,\,C\,],$$

where the intervals $[\,I,\,R\,]$ and $[\,0,\,C\,]$ are considered in the lattices G and L, respectively. Then, if we denote $c = C$ and $g = I$, we have $c \in L$, $g \in G$, $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/g$, which shows that L is strongly G-generated, as desired.

Latticial H-LT $\Longrightarrow$ Absolute H-LT

We show how the Absolute H-LT is an immediate consequence of the Latticial H-LT. Let $\mathcal{G}$ be a Grothendieck category, and let U, X be Artinian objects of $\mathcal{G}$ such that X is strongly U-generated (this means that each subobject of X is U-generated). We are going to prove that X is Noetherian.

Set $G := \mathcal{L}(U)$ and $L := \mathcal{L}(X)$. Then G and L are both Artinian upper continuous modular lattice. We have to prove that L is a Noetherian lattice. By Theorem 4.3.8, it is sufficient to check that L is strongly G-generated. To do that, let $a = A < B = b$ in $L = \mathcal{L}(X)$. Because B is U-generated by hypothesis, there exists a morphism $\alpha : U \longrightarrow B$ in $\mathcal{G}$ such that $\mathrm{Im}\,(\alpha) \nleqslant A$. But $\mathrm{Im}\,(\alpha) \simeq U/\mathrm{Ker}\,(\alpha)$, so, if we set $c := \mathrm{Im}\,(\alpha)$ and $k := \mathrm{Ker}\,(\alpha)$, then we have $c \leqslant b$, $c \nleqslant a$, and $c/0 \simeq 1/k$, which shows exactly that the lattice L is strongly G-generated.

In particular if U is an Artinian generator of $\mathcal{G}$, then any Artinian object $X \in \mathcal{G}$ is Noetherian, which is exactly the Absolute H-LT

Absolute H-LT $\Longrightarrow$ Relative H-LT

We are going to show how the Relative H-LT can be deduced from the Absolute H-LT. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R. Assume that R is τ-Artinian ring, and let M_R be a τ-Artinian module. We pass from Mod-R to the Grothendieck category Mod-$R/\mathcal{T}$ with the use of the canonical functor T_τ : Mod-$R \longrightarrow$ Mod-$R/\mathcal{T}$. Since R_R is a generator of Mod-R and T_τ is an exact functor that commutes with direct sums we deduce that $T_\tau(R)$ is a generator of Mod-$R/\mathcal{T}$, which is Artinian by Proposition 4.2.3. Now, again by Proposition 4.2.3, $T_\tau(M)$ is an Artinian object of Mod-$R/\mathcal{T}$, so, it is also Noetherian by the Absolute H-LT, i.e., M is τ-Noetherian, and we are done.

Relative H-LT $\Longrightarrow$ Absolute H-LT

We prove that the Absolute H-LT is a consequence of the Relative H-LT. Let $\mathcal{G}$ be a Grothendieck category having an Artinian generator U. Set $R_U := \mathrm{End}_{\mathcal{G}}(U)$, and let

$$S_U = \mathrm{Hom}_{\mathcal{G}}(U, -) : \mathcal{G} \longrightarrow \text{Mod-}R_U \quad \text{and} \quad T_U : \text{Mod-}R_U \longrightarrow \mathcal{G}$$

be the pair of functors from the Gabriel-Popescu Theorem setting, described in Section 3.3 just after Theorem 3.3.3. Then $T_U \circ S_U \simeq 1_{\mathcal{G}}$ and

$$\mathrm{Ker}(T_U) := \{\, M \in \text{Mod-}R_U \mid T_U(M) = 0\}$$

is a localizing subcategory of Mod-R_U. Let τ_U be the hereditary torsion theory (uniquely) determined by the localizing subcategory $\mathcal{T}_U := \mathrm{Ker}(T_U)$ of Mod-R_U. By the Gabriel-Popescu Theorem we have

$$\mathcal{G} \simeq \text{Mod-}R_U/\mathcal{T}_U \quad \text{and} \quad U \simeq (T_U \circ S_U)(U) = T_U(S_U(U)) = T_U(R_U).$$

Since U is an Artinian object of $\mathcal{G}$, so is also $T_U(R_U)$, which implies, by Proposition 4.2.3, that R_U is a τ_U-Artinian ring.

Now, let $X \in \mathcal{G}$ be an Artinian object of $\mathcal{G}$. Then, there exists a right R_U-module M such that $X \simeq T_U(M)$, so $T_U(M)$ is an Artinian object of $\mathcal{G}$, i.e., M is a τ_U-Artinian module. By the Relative H-LT, M is τ_U-Noetherian, so, again by Proposition 4.2.3, $X \simeq T_U(M)$ is a Noetherian object of $\mathcal{G}$, as desired.

The Faith's Δ-Σ version of the Relative H-LT

Recall that an injective module Q_R is said to be Σ-*injective* if any direct sum of copies of Q is injective. This concept is related with the concept of a τ-Noetherian module as follows.

Let Q_R be an injective module, and denote $\mathcal{T}_Q := \{ M_R \mid \mathrm{Hom}_R(M, Q) = 0\}$. Then $\mathcal{T}_Q$ is a localizing subcategory of Mod-R, and let τ_Q be the hereditary torsion theory on Mod-R (uniquely) determined by $\mathcal{T}_Q$. Note that for any hereditary torsion theory τ on Mod-R there exists an injective module Q_R such that $\tau = \tau_Q$.

A renowned theorem of Faith (1966) says that an injective module Q_R is Σ-injective if and only if R_R is τ_Q-Noetherian, or equivalently, if R satisfies the ACC on annihilators of subsets of Q. In order to uniformize the notation, Faith [49] introduced the concept of a Δ-*injective module* as being an injective module Q such that R_R is τ_Q-Artinian, or equivalently, R satisfies the DCC on annihilators of subsets of Q. Thus, the Relative H-LT is equivalent with the following *Faith's* Δ-Σ *version* of it.

THEOREM 4.4.1. *Any* Δ-*injective module is* Σ-*injective.*

PROOF. See [49, p. 3]. □

The Faith's counter version of the Relative H-LT

Let M_R be a module, and let $S := \mathrm{End}_R(M)$. Then M becomes a left S-module, and the module ${}_SM$ is called the *counter module* of M_R. We say that M_R is *counter-Noetherian* (respectively, *counter-Artinian*) if ${}_SM$ is a Noetherian (respectively, Artinian) module.

The next result is an equivalent version, in terms of counter modules, of the Relative H-LT.

THEOREM 4.4.2. *If* Q_R *is an injective module which is counter-Noetherian, then* Q_R *is counter-Artinian.*

PROOF. See [49, Theorem 7.1]. □

Absolute H-LT $\Longleftrightarrow$ Classical H-LT

Grothendieck categories having an Artinian generator are very special in view of the following surprising result of Năstăsescu.

THEOREM 4.4.3. *A Grothendieck category* $\mathcal{G}$ *has an Artinian generator if and only if* $\mathcal{G} \simeq \mathrm{Mod}\text{-}A$, *with* A *a right Artinian ring with identity.*

PROOF. See [73, Théorème 3.3]. □

Note that a heavy artillery has been used in the original proof of Theorem 4.4.3, namely: the Gabriel-Popescu Theorem, the Relative H-LT, as well as structure theorems for Δ-injective and Δ^*-projective modules. The Σ^*-*projective* and Δ^*-*projective* modules, introduced and investigated in [**72**], [**73**], are in a certain sense dual to the notions of Σ-injective and Δ-injective modules.

A more general result whose original proof is direct, without involving the many facts listed above, is the following.

THEOREM 4.4.4. *Let $\mathcal{G}$ be a Grothendieck category having a (finitely generated) generator U such that* $\mathrm{End}_{\mathcal{G}}(U)$ *is a right perfect ring. Then $\mathcal{G}$ has a (finitely generated) projective generator.*

PROOF. See [**38**, Theorem 2.2]. $\square$

Observe now that if $\mathcal{G}$ has an Artinian generator U, then, by the Absolute H-LT, U is also Noetherian, so, an object of finite length. Then $S = \mathrm{End}_{\mathcal{G}}(U)$ is a semiprimary ring, in particular it is right perfect. Now, by Theorem 4.4.4, $\mathcal{G}$ has a finitely generated projective generator, say P. If $A = \mathrm{End}_{\mathcal{G}}(P)$ then A is a right Artinian ring, and $\mathcal{G} \simeq$ Mod-A, which shows how Theorem 4.4.3 is an immediate consequence of Theorem 4.4.4.

REMARK. If $\mathcal{G}$ is a Grothendieck category having an Artinian generator U, then the right Artinian ring A in Theorem 4.4.3 for which $\mathcal{G} \simeq$ Mod-A is far from being the endomorphism ring of U, and does not seem to be canonically associated with $\mathcal{G}$. The existence of a right Artinian ring B, canonically associated with $\mathcal{G}$ and such that $\mathcal{G} \simeq$ Mod-B, is an easy consequence of a more general and more sophisticated construction in [**64**] of the *basic ring* of an arbitrary locally Artinian Grothendieck category. $\square$

Clearly Relative H-LT $\Longrightarrow$ Classical H-LT by taking as τ the hereditary torsion theory $(0, \text{Mod-}R)$ on Mod-R, and Absolute H-LT $\Longrightarrow$ Classical H-LT by taking as $\mathcal{G}$ the category Mod-R.

We conclude that the following implications between the various aspects of the H-LT discussed so far hold:

$$\boxed{\textbf{Latticial H-LT} \Longrightarrow \textbf{Relative H-LT} \Longleftrightarrow \textbf{Absolute H-LT} \Longleftrightarrow \textbf{Classical H-LT}}$$

$$\boxed{\textbf{Faith's } \boldsymbol{\Delta}\textbf{-}\boldsymbol{\Sigma} \textbf{ Theorem} \Longleftrightarrow \textbf{Relative H-LT} \Longleftrightarrow \textbf{Faith's counter Theorem}}$$

The Absolute and Relative Dual H-LT

Remember that the Absolute H-LT states that if $\mathcal{G}$ is a Grothendieck category with an Artinian generator, then any Artinian object of $\mathcal{G}$ is necessarily Noetherian, so it is natural to ask whether its dual holds, that is:

PROBLEM. (ABSOLUTE DUAL H-LT). *If $\mathcal{G}$ is a Grothendieck category having a Noetherian cogenerator, then does it follow that any Noetherian object of $\mathcal{G}$ is Artinian?* $\square$

Note that the Absolute Dual H-LT fails even for a module category Mod-R. To see this, let k be a universal differential field of characteristic zero with derivation D; then, the *Cozzens domain* $R = k[y, D]$ of differential polynomials over k in the derivation D is a principal right ideal domain which has a simple injective cogenerator S. So, $C = R \oplus S$ is both a Noetherian generator and cogenerator of Mod-R, which is clearly not Artinian (see [**2**, Section 4]).

However the Absolute Dual H-LT holds for large classes of Grothendieck categories, namely for the so called *commutative Grothendieck categories*, introduced in [**24**]. A Grothendieck category $\mathcal{C}$ is said to be *commutative* if there exists a commutative ring A with identity such that $\mathcal{G} \simeq \text{Mod-}A/\mathcal{T}$ for some localizing subcategory $\mathcal{T}$ of Mod-A. These are exactly those Grothendieck categories $\mathcal{G}$ having at least a generator U with a commutative ring of endomorphisms.

Recall that an object G of a Grothendieck category $\mathcal{G}$ is a *generator* of $\mathcal{G}$ if every object X of $\mathcal{G}$ is an epimorphic image $G^{(I)} \twoheadrightarrow X$ of a direct sum of copies of G for some set I. Dually, an object $C \in \mathcal{G}$ is said to be a *cogenerator* of $\mathcal{G}$ if every object X of $\mathcal{G}$ can be embedded $X \rightarrowtail C^I$ into a direct product of copies of C for some set I (see also Section 3.1 for the concepts of generator and cogenerator in an arbitrary category).

THEOREM 4.4.5. *The following assertions are equivalent for a commutative Grothendieck category* $\mathcal{G}$.

(1) $\mathcal{G}$ *has a Noetherian cogenerator.*
(2) $\mathcal{G}$ *has an Artinian generator.*
(3) $\mathcal{G} \simeq \text{Mod-}A$ *for some commutative Artinian ring with identity.*

PROOF. See [**2**, Theorem 3.2]. □

An immediate consequence of Theorem 4.4.5 is the following.

THEOREM 4.4.6. (ABSOLUTE DUAL HL-T). *If* $\mathcal{G}$ *is any commutative Grothendieck category having a Noetherian cogenerator, then every Noetherian object of* $\mathcal{G}$ *is Artinian.* □

If $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on Mod-R, then a module C_R is said to be a τ*-cogenerator* of Mod-R if $C \in \mathcal{F}$ and every module in $\mathcal{F}$ is cogenerated by C. The next result is the relative version of the Absolute Dual H-LT.

THEOREM 4.4.7. (RELATIVE DUAL HL-T). *Let R be a commutative ring with identity, and let* τ *be a hereditary torsion theory on* Mod-R *such that* Mod-R *has a* τ*-Noetherian* τ*-cogenerator. Then every* τ*-Noetherian R-module is* τ*-Artinian.* □

A Krull dimension-like extension of the Absolute H-LT

If $\mathcal{G}$ is a Grothendieck category and X is an object of $\mathcal{G}$, then recall that the *Krull dimension* of X, denoted by $k(X)$, is defined as $k(X) := k(\mathcal{L}(X))$, where $\mathcal{L}(X)$ is the lattice of all subobjects of X.

The definition of the Krull dimension of an object in a Grothendieck category $\mathcal{G}$ can also be given using a transfinite sequence of Serre subcategories of $\mathcal{G}$ and suitable quotient categories of $\mathcal{G}$ (see [**55**, Proposition 1.5]). Using this approach, the following extension of the Absolute H-LT has been proved:

THEOREM 4.4.8. *Let $\mathcal{G}$ be a Grothendieck category, and let U be a generator of $\mathcal{G}$ such that $k(U) = \alpha+1$ for some ordinal $\alpha \geqslant -1$. Then, for every object X of $\mathcal{G}$ having Krull dimension and for every ascending chain*

$$X_1 \leqslant X_2 \leqslant \ldots \leqslant X_n \leqslant \ldots$$

of subobjects of X, $\exists\, m \in \mathbb{N}$ such that $k(X_{i+1}/X_i) \leqslant \alpha$, $\forall\, i \geqslant m$.

PROOF. See [23, Theorem 3.1]. □

Note that for $\alpha = -1$ we obtain exactly the Absolute H-LT, because in this case, $X \in \mathcal{G}$ has Krull dimension if and only if $k(X) \leqslant 0$, i.e., if and only if X is Artinian.

It seems that the above result is really a *categorical property* of Grothendieck categories. As we already stressed before, the natural frame for the H-LT and its various extensions is *Lattice Theory*, being concerned as it is with descending and ascending chains in certain lattices, and therefore we shall present in the next subsection a very general version of Theorem 4.4.8 for upper continuous modular lattices.

A Krull dimension-like extension of the Latticial H-LT

In order to present an extension of Theorem 4.4.8 to lattices, which, on one hand, is interesting in its own right, and, on the other hand, provides another proof of it, avoiding the use of quotient categories, we need first a latticial substitute for the notion of generator of a Grothendieck category, which has been already presented in Section 4.3.

THEOREM 4.4.9. *Let L and G be upper continuous modular lattices. Suppose that $k(G) = \alpha + 1$ for some ordinal $\alpha \geqslant -1$ and L is strongly generated by G. If L has Krull dimension, then $k(L) \leqslant \alpha + 1$, and for every ascending chain*

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots$$

of elements of L, $\exists\, m \in \mathbb{N}$ such that $k(x_{i+1}/x_i) \leqslant \alpha$, $\forall\, i \geqslant m$.

PROOF. See [30, Theorem 3.16] for a thorough proof and [10] for a sketch of it. □

Two main ingredients are used in the proof of Theorem 4.4.9, namely the *Latticial H-LT* and a *localization* technique for modular lattices developed in [30] analogously with that for Grothendieck categories. In the next subsection we shall briefly discuss this technique.

Localization of modular lattices

The terminology and notation below are taken from the localization theory in Grothendieck categories. First, in analogy with the notion of a Serre subcategory of an Abelian category, we present below, as in [29], the notion of a *Serre class of lattices.*

DEFINITION. *By an* abstract class of lattices *we mean a non-empty subclass $\mathcal{X}$ of the class $\mathcal{M}_{0,1}$ of all modular lattices with 0 and 1, which is closed under lattice isomorphisms (i.e., if $L, K \in \mathcal{M}_{0,1}$, $K \simeq L$ and $L \in \mathcal{X}$, then $K \in \mathcal{X}$).*

We say that a subclass $\mathcal{X}$ of $\mathcal{M}_{0,1}$ is a Serre class for $L \in \mathcal{M}_{0,1}$ *if $\mathcal{X}$ is an abstract class of lattices, and for all $a \leqslant b \leqslant c$ in L, $c/a \in \mathcal{X}$ if and only if $b/a \in \mathcal{X}$ and $c/b \in \mathcal{X}$. A* Serre class of lattices *is an abstract class of lattices which is a Serre class for all lattices $L \in \mathcal{M}_{0,1}$.* □

Let $\mathcal{X}$ be an arbitrary non-empty subclass of $\mathcal{M}_{0,1}$ and let $L \in \mathcal{M}_{0,1}$ be a lattice. Define a relation $\sim_{\mathcal{X}}$ on L by:

$$a \sim_{\mathcal{X}} b \iff (a \vee b)/(a \wedge b) \in \mathcal{X}.$$

Then $\sim_{\mathcal{X}}$ is a congruence on L if and only if $\mathcal{X}$ is a Serre class for L. Recall that a *congruence* on a lattice L is an equivalence relation $\sim$ on L such that for all $a, b, c \in L$, $a \sim b$ implies $a \vee c \sim b \vee c$ and $a \wedge c \sim b \wedge c$. It is well-known that in this case the quotient set $L/\sim$ has a natural lattice structure, and the canonical mapping $L \longrightarrow L/\sim$ is a lattice morphism. If $\mathcal{X}$ is a Serre class for $L \in \mathcal{M}_{0,1}$, then the lattice $L/\sim_{\mathcal{X}}$ is called the *quotient lattice of* L *by* (or *modulo*) $\mathcal{X}$.

We define now for any non-empty subclass $\mathcal{X}$ of $\mathcal{M}_{0,1}$ and for any lattice L, a certain subset $\mathrm{Sat}_{\mathcal{X}}(L)$ of L, called the $\mathcal{X}$-*saturation* of L:

$$\mathrm{Sat}_{\mathcal{X}}(L) := \{\, x \in L \mid x \leqslant y \in L,\ y/x \in \mathcal{X} \implies x = y \,\}.$$

This is the precise analogue of the subset $\mathrm{Sat}_{\tau}(M) = \{\, N \leqslant M_R \,|\, M/N \in \mathcal{F} \,\}$ of the lattice $\mathcal{L}(M_R)$ of all submodules of a given module M_R, where $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on Mod-R.

Definition. *Let $\mathcal{X}$ be an arbitrary non-empty subclass of $\mathcal{M}_{0,1}$. We say that a lattice $L \in \mathcal{M}_{0,1}$ has an* $\mathcal{X}$-saturation *if there exists a mapping, called the* $\mathcal{X}$-saturation *of L*

$$L \longrightarrow \mathrm{Sat}_{\mathcal{X}}(L)\,, \quad x \longmapsto \overline{x},$$

satisfying the following two conditions:

(1) $x \leqslant \overline{x}$ *and* $\overline{x}/x \in \mathcal{X}$ *for all* $x \in L$.
(2) $x \leqslant y$ *in* $L \implies \overline{x} \leqslant \overline{y}$. □

If $\mathcal{X}$ is a Serre class for $L \in \mathcal{M}_{0,1}$ such that L has an $\mathcal{X}$-saturation $x \longmapsto \overline{x}$, and if we define

$$x \overline{\vee} y := \overline{x \vee y},\ \forall\, x,\, y \in \mathrm{Sat}_{\mathcal{X}}(L),$$

then the reader can easily check that $\mathrm{Sat}_{\mathcal{X}}(L)$ becomes a modular lattice with respect to $\leqslant$, $\wedge$, $\overline{\vee}$, $\overline{0}$, 1.

By Proposition 4.2.3, for any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on Mod-R and any module M_R, the lattice $\mathrm{Sat}_{\tau}(M)$ is isomorphic to the lattice $\mathcal{L}(T_{\tau}(M))$ of all subobjects of the object $T_{\tau}(M)$ in the quotient category Mod-$R/\mathcal{T}$, where

$$T_{\tau} : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

is the canonical functor. The same happens also in our latticial frame: if $\mathcal{X}$ is a Serre class for $L \in \mathcal{M}_{0,1}$ such that L has an $\mathcal{X}$-saturation, then

$$L/\sim_{\mathcal{X}} \simeq \mathrm{Sat}_{\mathcal{X}}(L).$$

Consequently, the lattice L is $\mathcal{X}$-Noetherian (respectively, $\mathcal{X}$-Artinian) $\iff$ the lattice $\mathrm{Sat}_{\mathcal{X}}(L)$ is Noetherian (respectively, Artinian) $\iff$ the lattice $L/\sim_{\mathcal{X}}$ is Noetherian (respectively, Artinian).

If $\mathcal{X}$ is a Serre class of lattices for a lattice L, one may define as in [29] the relative conditions $(\mathcal{E})_{\mathcal{X}}$ and $(\mathcal{BL})_{\mathcal{X}}$ in order to prove the following Latticial H-LT relative to $\mathcal{X}$.

Theorem 4.4.10. (Relative Latticial H-LT). *Let $\mathcal{X} \subseteq \mathcal{M}_{0,1}$ be a Serre class for a lattice $L \in \mathcal{M}_{0,1}$ such that L has an $\mathcal{X}$-saturation and L is $\mathcal{X}$-Artinian. Then L is $\mathcal{X}$-Noetherian if and only if L satisfies both conditions $(\mathcal{E})_{\mathcal{X}}$ and $(\mathcal{BL})_{\mathcal{X}}$.*

Proof. See [29, Theorem 4.9] □

Serre classes of lattices which are closed under taking arbitrary joins, we next introduce, are called *localizing classes of lattices* and they play the same role as that of localizing subcategories in the setting of Grothendieck categories. More precisely, we have the following:

Definition. *Let* $\mathcal{X}$ *be a non-empty subclass of* $\mathcal{M}_{0,1}$ *and let* L *be a complete modular lattice. We say that* $\mathcal{X}$ *is a* localizing class for L *if* $\mathcal{X}$ *is a Serre class for* L*, and for any* $x \in L$ *and for any family* $(x_i)_{i\in I}$ *of elements of* $1/x$ *such that* $x_i/x \in \mathcal{X}$ *for all* $i \in I$*, we have* $(\bigvee_{i\in I} x_i)/x \in \mathcal{X}$*. By a* localizing class of lattices *we mean a Serre class of lattices which is a localizing class for every complete modular lattice.* □

Note that if $\mathcal{X}$ is a localizing class for a complete modular lattice L then L has an $\mathcal{X}$-saturation, which is uniquely determined. For more details on localization of modular lattices, the reader is referred to [29], [30], and [32].

Three open questions

(1) *If* R *is any ring with right Krull dimension, is it true that* $k^0(R) \leqslant k(R)$?
This question has been raised by Albu and Smith in 1991, and also mentioned in [31, Question 1]. Observe that the answer is *yes* for $k(R) = 0$, which is exactly the Classical H-LT. Other cases when the answer is *yes*, according to [31], are when R is one of the following types of rings:

- a commutative Noetherian ring, or
- a commutative ring with Krull dimension 1, or
- a commutative domain with Krull dimension 2, or
- a valuation domain with Krull dimension, or
- a right Noetherian right V-ring.

(2) Similarly with the *right global homological dimension* of a ring R, two kinds of "global dimension" related to the Krull dimension and dual Krull dimension of a ring R have been defined in [31]: the *right global Krull dimension* r.gl.$k(R)$ and the *right global dual Krull dimension* r.gl.$k^0(R)$ of a ring, as being the supremum of $k(M_R)$ and $k^0(M_R)$, respectively, when M_R is running in the class of all modules having Krull dimension.
Similarly with Question 1, one may ask:

What is the order relation between r.gl.$k(R)$ *and* r.gl.$k^0(R)$*?*

Note that, though according to [36, Corollary 1.3], $k^0(R) \leqslant k(R)$ for any *valuation ring* R having Krull dimension, unexpectedly one has the opposite order relation r.gl.$k(R) \leqslant$ r.gl.$k^0(R)$ for any such ring R by [36, Theorem 2.4]. Recall that a valuation ring is a commutative ring with identity whose ideals are totally ordered by inclusion.

(3) *Does the result of Theorem 4.4.9 fail when* $k(G)$ *is a limit ordinal?*
We suspect that the answer is *yes*, even in the module case.

CHAPTER 5

THE OSOFSKY-SMITH THEOREM

In this chapter we discuss various aspects of the following renowned result of Module Theory giving sufficient conditions for a finitely generated (respectively, cyclic) module to be a finite direct sum of uniform submodules.

The Osofsky-Smith Theorem (O-ST). ([79, Theorem 1]). *A finitely generated (respectively, cyclic) right R-module such that all of its finitely generated (respectively, cyclic) subfactors are CS modules is a finite direct sum of uniform submodules.* $\square$

Recall that a module M is said to be *CS* (or *extending*) if every submodule of M is essential in a direct summand of M, or, equivalently, any complement submodule of M is a direct summand of M. By subfactor of M one understands any submodule of a factor module of M. Recall that in Module Theory one says that a submodule N of M is a *complement* if there exists a submodule L of M such that $N \cap L = 0$ and N is maximal in the set of all submodules P of M such that $P \cap L = 0$, i.e., the element N of the lattice $\mathcal{L}(M)$ of all submodules of M is a pseudo-complement element in this lattice. The name CS is an acronym for Complements submodules are direct Summands. More about CS modules can be found in the monograph [48], entirely devoted to them.

Though the Osofsky-Smith Theorem is a module-theoretical result, our contention is that it is a result of a strong latticial nature. In this chapter a latticial version of this theorem is presented, and applications to Grothendieck categories and module categories equipped with a torsion theory are given.

5.1. CC lattices

The purpose of this section is to discuss CC lattices, introduced in [21] as the latticial counterparts of CS modules. These are exactly the lattices satisfying the first condition (C_1) from the list below of five conditions (C_i), $i = 1, 2, 3, 11, 12$. We also present the concept of a CEK lattice needed in the next section.

Throughout this section L will denote a modular lattice with a least element 0 and a greatest element 1, i.e., $L \in \mathcal{M}_{0,1}$.

The conditions (C_i) for lattices

Recall that for any lattice L we introduced in Chapters 1 and 2 the following notation:

$P(L) :=$ the set of all *pseudo-complement* elements of L (P for "*Pseudo*"),
$E(L) :=$ the set of all *essential* elements of L (E for "*Essential*"),
$C(L) :=$ the set of all *closed* elements of L (C for "*Closed*"),
$D(L) :=$ the set of all *complement* elements of L (D for "*Direct summand*"),

$K(L) :=$ the set of all *compact* elements of L (K for "*Kompakt*").

We present now five conditions (C_i), $i = 1, 2, 3, 11, 12$, introduced in [**21**] as the latticial counterparts of the well-known corresponding conditions in Module Theory.

Definitions. *For a lattice L one may consider the following conditions:*

(C_1) *For every $x \in L$ there exists $d \in D(L)$ such that $x \in E(d/0)$.*
(C_2) *For every $x \in L$ such that $x/0 \simeq d/0$ for some $d \in D(L)$, one has $x \in D(L)$.*
(C_3) *For every $d_1, d_2 \in D(L)$ with $d_1 \wedge d_2 = 0$, one has $d_1 \vee d_2 \in D(L)$.*
(C_{11}) *For every $x \in L$ there exists a pseudo-complement p of x with $p \in D(L)$.*
(C_{12}) *For every $x \in L$ there exist $d \in D(L)$, $e \in E(d/0)$, and a lattice isomorphism $x/0 \simeq e/0$.* □

Definitions. *A lattice L is called* CC *or* extending *if it satisfies (C_1),* continuous *if it satisfies (C_1) and (C_2), and* quasi-continuous *if it satisfies (C_1) and (C_3).* □

CC lattices

We present now some characterizations and basic properties of CC lattices.

Lemma 5.1.1. *The following assertions hold for a lattice L.*

(1) *$D(L) \subseteq P(L) \subseteq C(L)$.*
(2) *$D(L) \cap (a/0) \subseteq D(a/0)$ for every $a \in L$.*
(3) *$D(L) \cap (d/0) = D(d/0)$ for every $d \in D(L)$.*

Proof. (1) Let $d \in D(L)$. Then there exists $c \in L$ with $c \vee d = 1$ and $c \wedge d = 1$. If $c' \in L$ is such that $c \leqslant c'$ and $d \wedge c' = 0$, then, by modularity we have

$$c' = 1 \wedge c' = (c \vee d) \wedge c' = c \vee (d \wedge c') = c \vee 0 = c,$$

which proves that $d \in P(L)$, and so $D(L) \subseteq P(L)$. The other inclusion $P(L) \subseteq C(L)$ follows from Proposition 1.2.16.

(2) Let $d \in D(L) \cap (a/0)$, and let $c \in L$ be a complement of d in L. Then $1 = c \vee d$ and $c \wedge d = 0$. It follows that $(c \wedge a) \wedge d = 0$ and $(c \wedge a) \vee d = (c \vee d) \wedge a = 1 \wedge a = a$, which shows that $c \wedge a$ is a complement of d in $a/0$, i.e., $d \in D(a/0)$.

(3) Let $d' \in D(d/0)$. Then there exists $d'' \in L$ such that $d' \wedge d'' = 0$ and $d' \vee d'' = d$. Also, $a \vee d = 1$ and $a \wedge d = 0$ for some $a \in L$. Thus $d' \vee (d'' \vee a) = 1$. Now, observe that $a \wedge (d' \vee d'') = a \wedge d = 0$, so we can apply Lemma 1.2.6 to obtain $d' \wedge (d'' \vee a) = 0$. This shows that $d'' \vee a$ is a complement of d' in L, i.e., $d' \in D(L)$. Since $d' \leqslant d$, we deduce that $d' \in D(L) \cap (d/0)$. So $D(d/0) \subseteq D(L) \cap (d/0)$. The other inclusion follows from (2). □

The next result explains the term of a CC lattice, acronym for *C*losed elements are *C*omplements.

Proposition 5.1.2. *The following statements hold for a lattice L.*

(1) *L is uniform $\Longrightarrow$ L is CC, and, if additionally L is indecomposable, then the inverse implication "$\Longleftarrow$" also holds.*
(2) *If additionally L is essentially closed (in particular, if L is upper continuous) then*

$$L \text{ is CC} \Longleftrightarrow C(L) \subseteq D(L) \Longleftrightarrow C(L) = D(L).$$

(3) *If additionally* L *is strongly pseudo-complemented* (*in particular, if* L *is upper continuous*) *then*

$$L \text{ is } CC \iff C(L) \subseteq D(L) \iff C(L) = D(L) \iff$$
$$\iff P(L) \subseteq D(L) \iff P(L) = D(L).$$

Proof. (1) Assume that L is uniform, and let $x \in L$. If $x = 0$ then $0 \in D(L)$ and $0 \in E(0/0)$. If $x \neq 0$ then $1 \in D(L)$ and $x \in E(1/0) = E(L)$. So L is CC.

Now assume that L is CC and indecomposable, and let $0 \neq x \in L$. By hypothesis, $x \in E(d/0)$ for some $d \in D(L) = \{0, 1\}$, so necessarily $d = 1$, and then $x \in E(1/0) = E(L)$. Hence L is uniform.

(2) Assume that L is CC, and let $x \in C(L)$. Then, there exists $d \in D(L)$ such that $x \in E(d/0)$, and hence $x = d \in D(L)$ because L is essentially closed. So $C(L) \subseteq D(L)$. Observe that, by Lemma 5.1.1(1), $C(L) \subseteq D(L) \iff C(L) = D(L)$.

Finally assume that $C(L) \subseteq D(L)$, and let $x \in L$. There exists $c \in C(L)$ such that $x \in E(c/0)$. By assumption, $c \in D(L)$. It follows that L is CC.

(3) follows at once from (2), Theorem 1.2.24, and Corollary 1.2.17. □

Proposition 5.1.3. *Let* L *be a strongly pseudo-complemented lattice* (*in particular an upper continuous lattice*). *If* L *is a CC lattice then so is also* $d/0$ *for any* $d \in D(L)$, *in other words, the CC condition is inherited by complement intervals.*

Proof. Assume that L is CC, and let $c \in C(d/0)$. Since $d \in D(L) \subseteq C(L)$ by Lemma 5.1.1(1), it follows that $c \in C(L)$ by Corollary 1.2.14. But $C(L) = D(L)$ by Proposition 5.1.2, therefore $c \in D(L) \cap (d/0) = D(d/0)$ by Lemma 5.1.1(3). Thus $C(d/0) \subseteq D(d/0)$. Now, observe that $d/0$ is strongly pseudo-complemented by Lemma 1.2.20, so we can apply again Proposition 5.1.2(3) to deduce that $d/0$ is a CC lattice. □

Corollary 5.1.4. *Let* L *be a strongly pseudo-complemented CC lattice. Then* L *has finite Goldie dimension if and only if* 1 *is a finite direct join of uniform elements of* L.

Proof. One implication is clear. For the other one, assume that L has finite Goldie dimension. Then L contains a uniform element v. Let $c \in C(L)$ be such that $v \in E(c/0)$. Then $c \in D(L)$ because L is CC, so $1 = c \dot{\vee} c'$, for some $c' \in L$. It follows that $c'/0$ is also CC by Proposition 5.1.3. Now observe that $u(c'/0) < u(L)$ and $c'/0$ is strongly pseudo-complemented, so the proof proceeds by induction on $u(L)$. □

CEK lattices

We discuss the concept of a *CEK* lattice, that will be necessary in proving a key lemma used in the proof of the main result of this chapter.

Definitions. *Let* L *be a lattice.*

(1) *An element* $a \in L$ *is called* essentially compact *if* $E(a/0) \cap K(L) \neq \varnothing$. *We denote by* $E_k(L)$ *the set of all essentially compact elements of* L.

(2) L *is called CEK* (*for* Closed *are* Essentially Compact) *if every closed element of* L *is essentially compact, i.e.,* $C(L) \subseteq E_k(L)$. □

The next result provides large classes of CEK lattices.

Proposition 5.1.5. *Let L be a non-zero complete modular lattice having the following property:*

$(\dagger)$ *For every $0 \neq x \in L$ there exists $0 \neq k \in K(L)$ with $k \leqslant x$.*

In particular, L can be any compactly generated lattice.

Then L has finite Goldie dimension if and only if each element of L is essentially compact, i.e., $L = E_k(L)$. In particular, any modular lattice with finite Goldie dimension satisfying $(\dagger)$ is CEK.

Proof. Assume that L has finite Goldie dimension, and let $a \in L$. Then the interval $a/0$ has also finite Goldie dimension, so there exists an independent family $(u_i)_{1 \leqslant i \leqslant n}$ of uniform elements of $a/0$ such that $\bigvee_{1 \leqslant i \leqslant n} u_i \in E(a/0)$. By hypothesis, for every i, $1 \leqslant i \leqslant n$, there exist $0 \neq k_i \in K(L)$ with $k_i \leqslant u_i$. Then, if we set $k := \bigvee_{1 \leqslant i \leqslant n} k_i$ and $u := \bigvee_{1 \leqslant i \leqslant n} u_i$, we have $k \in E(u/0) \cap K(L)$, so $k \in E(a/0) \cap K(L)$, as desired.

Conversely, assume that L has infinite Goldie dimension. Then $L \setminus \{0\}$ contains an infinite independent set $\{ x_1, x_2, \ldots \}$. Since L is a complete lattice, we may consider the element $x := \bigvee_{i \in \mathbb{N}} x_i$. Then $x \notin E_k(L)$, for otherwise, it would exist $c \leqslant x$ such that $c \in E(x/0) \cap K(L)$. Then $c \leqslant \bigvee_{1 \leqslant i \leqslant m} x_i$ for some $m \in \mathbb{N}$, so

$$c \wedge x_{m+1} \leqslant \Big(\bigvee_{1 \leqslant i \leqslant m} x_i \Big) \wedge x_{m+1} = 0,$$

and then, $c \notin E(x/0)$, which is a contradiction. This means that $x \notin E_k(L)$, and we are done. □

5.2. The Latticial Osofsky-Smith Theorem

In this section we prove the latticial version of the module-theoretical Osofsky-Smith Theorem. Our contention is that the natural setting for this theorem and its various extensions is Lattice Theory, being concerned as it is, with latticial concepts like essential, uniform, complement, pseudo-complements elements, and direct joins in certain lattices.

Three lemmas

Lemma 5.2.1. *Let L be a compact, compactly generated, modular lattice. Assume that all compact intervals b/a of L are CEK, i.e., every $c \in C(b/a)$ is an essentially compact element of b/a. Then $D(L)$ is a Noetherian poset.*

Proof. See [**14**, Lemma 2.1] for a very technical 6-page proof. □

Observe that he condition that the lattice L is compact is necessary in Lemma 5.2.1. Indeed, let M be an infinite dimensional vector space over the field F, and let L denote the lattice $\mathcal{L}(M)$ of all submodules of ${}_F M$. Then all compact intervals b/a of L, i.e., all the lattices of all F-submodules of all finite dimensional quotient modules V/W with $W \leqslant V \leqslant {}_F M$ are CEK by Proposition 5.1.5, but $D(L)$ is not Noetherian.

The next result is the latticial counterpart of a well-known result asserting that a non-zero module M_R satisfying ACC or DCC on direct summands is a finite direct sum of finitely many indecomposable submodules (see, e.g., [**39**, Proposition 10.14]).

LEMMA 5.2.2. *Let $0 \neq L \in \mathcal{M}_{0,1}$ and assume that the set $D(L)$ of complement elements of L is either Noetherian or Artinian. Then 1 is a direct join of finitely many indecomposable elements of L.*

PROOF. Deny. Then L is not indecomposable, so we can write

$$1 = x_1 \dot{\vee} y_1$$

with $x_1, y_1 \in D(L) \setminus \{0,1\}$ such that y_1 cannot be written as a direct join of finitely many indecomposable elements of L. Then, we can write

$$y_1 = x_2 \dot{\vee} y_2$$

with $x_2, y_2 \in D(L) \setminus \{0,1\}$ such that y_2 cannot be written as a direct join of finitely many indecomposable elements of L, and so on.

Thus, we obtain the following infinite chains of elements of $D(L)$:

$$x_1 < x_1 \dot{\vee} x_2 < \dots \quad \text{and} \quad 1 > y_1 > y_2 > \dots ,$$

which is a contradiction. □

LEMMA 5.2.3. *Any modular, upper continuous, compact, CC lattice is CEK.*

PROOF. We have to show that $C(L) \subseteq E_k(L)$. By Proposition 5.1.2(2), this means that $D(L) \subseteq E_k(L)$. So, let $d \in D(L)$. Then $d \in E(d/0) \cap K(L)$ by Proposition 2.1.18(1), so L is CEK. □

The main result

THEOREM 5.2.4. (LATTICIAL O-ST). *Let L be a compact, compactly generated, modular lattice. Assume that all compact subfactors of L are CC. Then 1 is a finite direct join of uniform elements of L.*

PROOF. First, observe that the given lattice L being compactly generated, is also upper continuous. Recall that by a subfactor of L we mean any interval b/a of L. By assumption, every compact subfactor of L is CC, so CEK by Lemma 5.2.3. Using now Lemma 5.2.1, we deduce that $D(L)$ is a Noetherian poset, so, by Lemma 5.2.2, $1 = \dot{\bigvee}_{1 \leqslant i \leqslant n} d_i$ is a finite direct join of indecomposable elements d_i of L. Since L is CC, so is also any $d_i/0$ by Proposition 5.1.3. Finally, every d_i is uniform by Proposition 5.1.2(1), and we are done. □

Following [**47**], a right R-module M is said to be *CF* if every closed submodule of M is finitely generated, and *completely CF* provided every quotient of M is also CF. More generally, we say that a lattice L is *CK* (acronym for *C*losed are *K*ompact) if every closed element of L is compact, i.e., $C(L) \subseteq K(L)$. Clearly, any CK lattice is also CEK, so we deduce at once from Lemmas 5.2.1 and 5.2.2 the following result.

PROPOSITION 5.2.5. *Let L be a compact, compactly generated, modular lattice. Assume that all compact subfactors of L are CK. Then $D(L)$ is a Noetherian poset, in particular 1 is a finite direct join of indecomposable elements of L.* □

We extend now the Latticial O-ST, valid for any compact, compactly generated, modular lattice having all compact subfactors CC, to more general lattices, so that it can be also applied to cyclic modules (which have no latticial counterparts).

Denote by $\mathcal{K}$ the class of all compact lattices and by $\mathcal{U}$ the class of all upper continuous lattices, and let $\mathcal{P}$ be a non-empty subclass of $\mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the following three conditions:

- (P_1) If $L \in \mathcal{P}$, $L' \in \mathcal{L}$, and $L \simeq L'$ then $L' \in \mathcal{P}$.
- (P_2) If $L \in \mathcal{P}$ then $1/a \in \mathcal{P}$, $\forall\, a \in L$.
- (P_3) If $L \in \mathcal{P}$ and $b/a \in \mathcal{P}$ is a subfactor of L, then $\exists\, c \in L$ such that $c/0 \in \mathcal{P}$ and $b = a \vee c$.

Examples of classes $\mathcal{P}$ satisfying the conditions $(P_1) - (P_3)$ above are:

- any $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ such that $L \in \mathcal{P} \Longrightarrow (1/a \in \mathcal{P}\ \&\ a/0 \in \mathcal{P}, \forall\, a \in L)$;
- the class of all compact, compactly generated, modular lattices;
- the class of all compact, semi-atomic, upper continuous, modular lattices;
- the class of lattices isomorphic to lattices of all submodules of all cyclic right R-modules.

For any lattice L we set $\mathcal{P}(L) := \{\, c \in L \mid c/0 \in \mathcal{P}\,\}$. Observe that $\varnothing \neq \mathcal{P}(L) \subseteq K(L)$ if $L \in \mathcal{U}$.

Theorem 5.2.6. (Latticial $\mathcal{P}$-O-ST). *Let $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the conditions $(P_1) - (P_3)$ above, and let $L \in \mathcal{P}$. Assume that all subfactors of L in $\mathcal{P}$ are CC. Then 1 is a finite direct join of uniform elements of L.*

Proof. See [13, Theorem 3.7]. $\square$

Corollary 5.2.7. *Let $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the conditions $(P_1) - (P_3)$ above. Then, the following statements are equivalent for a complete modular lattice L such that any of its elements is a join of elements of $\mathcal{P}(L)$.*

(1) *L is semi-atomic.*
(2) *F is CC and $K(F) \subseteq D(F)$ for every subfactor $F \in \mathcal{P}$ of L.*

Proof. (1) $\Longrightarrow$ (2) As already mentioned in Section 2.1, any subfactor F of L is semi-atomic, so complemented. If follows that F is CC and $K(F) \subseteq F = D(F)$.

(2) $\Longrightarrow$ (1) Notice that, by hypothesis, L is a compactly generated lattice. Let $c \in K(L)$ with $C := c/0 \in \mathcal{P}$, in other words, $c \in \mathcal{P}(L)$. Then c is a finite direct join of uniform elements of L by Theorem 5.2.6 applied to C.

Let $d \leqslant c$ with $d \in D(C)$ and d uniform. Then, for every $0 \neq d' \leqslant d$ with $d' \in K(d/0)$ one has $d' \in D(d/0)$ by hypothesis, because, by (P_2) and (P_1), the subfactor $d/0$ of L is in $\mathcal{P}$. Since d is uniform, we deduce that $d' = d$, so $d \in K(d/0)$. Let $0 \neq b \leqslant d$, and let $0 \neq b' \leqslant b$ with $b' \in K(d/0)$. It follows that $b' \in D(d/0)$ and so, $d = b' \leqslant b \leqslant d$ because d is uniform. Thus, for any $0 \neq b \leqslant d$, one has $b = d$. Consequently, d is an atom of L, which implies that $C = c/0$ is a semi-atomic lattice. By hypothesis, 1 is a join of compact elements of L in $\mathcal{P}(L)$, so 1 is a join of atoms of L, i.e., L is a semi-atomic lattice, as desired. $\square$

Notice that Corollary 5.2.7 is a latticial version of the following module-theoretical result:

A right R-module M is semisimple $\Longleftrightarrow$ every cyclic subfactor of M is M-injective

(see [48, Corollary 7.14]), which, in turn, is a "modularization" of the well-known *Osofsky's Theorem* [78] saying that a ring R is semisimple if and only if every cyclic right R-module is injective.

Because we do not have in hand a good latticial substitute for the notion of an injective module, the result above seems to be the best latticial counterpart of the Osofsky's Theorem. However, using the concept of a linear morphism of lattices, recently introduced in [17] and briefly discussed in the next subsection, we expect to provide a consequence, involving linear injective lattices, of the Latticial Osofsky-Smith Theorem.

Linear lattice morphisms

The concept of a *linear morphism* of lattices we present below evokes the property of a linear mapping $\varphi : M \longrightarrow N$ between modules M_R and N_R to have a kernel $\mathrm{Ker}\,(\varphi)$ and to verify the Fundamental Theorem of Isomorphism $M/\mathrm{Ker}\,(\varphi) \simeq \mathrm{Im}\,(\varphi)$.

DEFINITION. *Let $f : L \longrightarrow L'$ be a mapping between the lattice L with least element 0 and last element 1 and the lattice L' with least element $0'$ and last element $1'$.*

The mapping f is said to be a linear morphism *if there exist $k \in L$, called a* kernel *of f, and $a' \in L'$ such that the following two conditions are satisfied.*

(1) $f(x) = f(x \vee k)$, $\forall\, x \in L$.
(2) f *induces a lattice isomorphism* $\bar{f} : 1/k \xrightarrow{\sim} a'/0'$, $\bar{f}(x) = f(x)$, $\forall\, x \in 1/k$. □

If $\varphi : M_R \longrightarrow N_R$ is a morphism of modules, then the mapping

$$f : \mathcal{L}(M_R) \longrightarrow \mathcal{L}(N_R),\ f(X) := \varphi(X),\ \forall\, X \leqslant M,$$

is clearly a linear morphism of lattices with kernel $\mathrm{Ker}\,(\varphi)$.

We present now some of the basic properties of linear morphisms of lattices.

PROPOSITION 5.2.8. *Let $f : L \longrightarrow L'$ be a linear morphism of lattices with a kernel k. Then, the following assertions hold.*

(1) *For $x, y \in L$, $f(x) = f(y) \Longleftrightarrow x \vee k = y \vee k$.*
(2) *$f(k) = 0'$ and k is the greatest element of L having this property, so, the kernel of a linear morphism is uniquely determined.*
(3) *f commutes with arbitrary joins, i.e., $f\,(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$ for any family $(x_i)_{i \in I}$ of elements of L, provided both joins exist.*
(4) *f is an increasing mapping.*
(5) *f preserves intervals, i.e., for any $u \leqslant v$ in L, one has $f(v/u) = f(v)/f(u)$.*

PROOF. See [17, Proposition 1.3 and Corollary 1.4] and [18, Lemma 0.6]. □

PROPOSITION 5.2.9. *The following statements hold.*

(1) *The class $\mathcal{M}_{0,1}$ of all bounded modular lattices becomes a category, denoted by $\mathcal{LM}$, if for any $L, L' \in \mathcal{M}_{0,1}$ one takes as morphisms from L to L' all the linear morphisms from L to L'.*
(2) *The isomorphisms in the category $\mathcal{LM}$ are exactly the isomorphisms in the full category $\mathcal{M}_{0,1}$ of the category $\mathcal{L}_{0,1}$ of all bounded lattices.*
(3) *The monomorphisms (respectively, epimorphisms) in the category $\mathcal{LM}$ are exactly the injective (respectively, surjective) linear morphisms.*

(4) *The subobjects of* $L \in \mathcal{LM}$ *can be taken as the intervals* $a/0$ *for any* $a \in L$.

Proof. See [17, Proposition 2.2]. $\square$

According to the terminology in [86], a *quasi-frame,* or shortly, a *qframe* is nothing else than an upper continuous modular lattice, and a *qframe morphism* is by definition any mapping between two qframes that commutes with arbitrary joins and preserves intervals, i.e., satisfies the properties (3) and (5) in Proposition 5.2.8, respectively; further, a qframe morphism $f : L \longrightarrow L'$ is said to be *algebraic* if the restriction $h : 1/K(f) \longrightarrow L'$ of f to the interval $1/K(f)$ of L is injective, where $K(f) := \bigvee_{f(x)=0'} x$. If $f : L \longrightarrow L'$ is any linear morphism between two qframes then, by Proposition 5.2.8(2), $K(f)$ coincides with its kernel k we defined above.

The connection between the linear morphisms of lattices and qframe morphisms is established by the next result.

Proposition 5.2.10. *The following statements are equivalent for a mapping* $f : L \longrightarrow L'$ *between two qframes* L *and* L'.

(1) f *is a linear morphism.*
(2) f *is an algebraic qframe morphism.*

Proof. See [18, Proposition 0.8]. $\square$

The example below shows that a qframe morphism is not necessarily algebraic, i.e., a linear morphism. Consider the four-element lattice $L = \{0, a, b, 1\}$ with $0 < a, b < 1$ and a, b incomparable. Also, consider $L' = \{0', 1'\}$ with $0' < 1'$. Clearly, L and L' are both qframes. Let $f : L \longrightarrow L'$ be the map defined by

$$f(0) = 0' \text{ and } f(a) = f(b) = f(1) = 1'.$$

Then f is a qframe morphism with $K(f) = 0$. However, f is not a linear morphism and so, it is not an algebraic qframe morphism. Indeed, if f would be a linear morphism, then its kernel as a linear morphism would be $k = 0$, so $L = 1/k \simeq f(L) = L'$, which is a contradiction.

Linear injective lattices

Recall that a module Q_R is said to be M-*injective*, where M_R is another module, if for every submodule N of M, every morphism $N \longrightarrow Q$ can be extended to a morphism $M \longrightarrow Q$. If $\mathcal{A}$ is a non-empty class of right R-modules, then Q is called $\mathcal{A}$-*injective* if it is M-injective for every $M \in \mathcal{A}$.

When trying to obtain latticial counterparts of these module-theoretical concepts, there are at least two options, depending on what kind of morphisms are we taking into account: usual lattice morphisms or linear morphisms of lattices.

Definitions. *Let* $Q, L \in \mathcal{L}_{0,1}$. *The lattice* Q *is said to be* L-injective *if for every sublattice* S *of* L, *every lattice morphism* $S \longrightarrow Q$ *can be extended to a lattice morphism* $L \longrightarrow Q$.

The lattice Q *is said to be* linear L-injective *if for every element* $a \in L$, *every linear morphism* $a/0 \longrightarrow Q$ *can be extended to a linear morphism* $L \longrightarrow Q$.

If $\mathcal{C}$ *is a non-empty subclass of* $\mathcal{L}_{0,1}$, *then* Q *is said to be* $\mathcal{C}$-injective (*resp.* linear $\mathcal{C}$-injective) *if it is* L-injective (*resp. linear* L-injective) *for every* $L \in \mathcal{C}$.

The lattice Q *is called* injective (*resp.* linear injective) *if it is* $\mathcal{L}_{0,1}$*-injective* (*resp.* linear $\mathcal{L}_{0,1}$-injective). □

Note that the injective lattices are exactly the injective objects of the category $\mathcal{L}_{0,1}$ of all bounded lattices. If we restrict now our considerations from the class $\mathcal{L}_{0,1}$ to the subclass $\mathcal{M}_{0,1}$ of all bounded modular lattices, then, in view of Proposition 5.2.9, the linear injective lattices are precisely the injective objects of the category $\mathcal{LM}$.

By [**17**, Section 4], there are neither non-zero injective lattices nor non-zero linear injective lattices, but plenty of $\mathcal{C}$-injective lattices and linear $\mathcal{C}$-injective lattices; also, there are no connections between $\mathcal{C}$-injective lattices and linear $\mathcal{C}$-injective lattices.

Lattice preradicals

The concept of a linear morphism of lattices is the main ingredient in defining the latticial counterpart of the module-theoretical concept of *preradical.*

As in Section 4.4 where we defined the concept of a Serre class of lattices, we say that a non-empty subclass $\mathcal{C}$ of the class $\mathcal{M}_{0,1}$ of all bounded modular lattices is *hereditary* if it is an abstract class and for any $L \in \mathcal{M}_{0,1}$ and any $a \leqslant b \leqslant c$ in L such that $c/a \in \mathcal{C}$, it follows that $b/a \in \mathcal{C}$.

For any non-empty subclass $\mathcal{C}$ of $\mathcal{M}_{0,1}$ we shall denote by $\mathcal{LC}$ the full subcategory of $\mathcal{LM}$ having $\mathcal{C}$ as the class of its objects.

Definition. *Let* $\mathcal{C}$ *be a hereditary subclass of* $\mathcal{M}_{0,1}$. *A* lattice preradical *on* $\mathcal{C}$ *is any functor* $r : \mathcal{LC} \longrightarrow \mathcal{LC}$ *satisfying the following two conditions.*

(1) $r(L) \leqslant L$, *(i.e.,* $r(L)$ *is a subobject of* L*) for any* $L \in \mathcal{LC}$.

(2) *For any morphism* $f : L \longrightarrow L'$ *in* $\mathcal{LC}$, $r(f) : r(L) \longrightarrow r(L')$ *is the restriction and corestriction of* f *to* $r(L)$ *and* $r(L')$, *respectively.*

In other words, a lattice preradical is nothing else than a subfunctor of the identity functor $1_{\mathcal{LC}}$ *of the category* $\mathcal{LC}$. □

For example, the assignment $L \mapsto \mathrm{Soc}\,(L)/0$ defines a preradical on the full subcategory $\mathcal{LM}_u$ of $\mathcal{LM}$ consisting of all upper continuous modular lattices. Lattice preradicals, introduced in [**18**], preserve many of the properties of module preradicals; in particular, they commute with arbitrary direct joins (see [**18**, Proposition 1.4]).

5.3. The Categorical Osofsky-Smith Theorem

In this section we deal with the *absolutization* of the module-theoretical O-ST. Thus, by applying the Latticial O-ST to the specific case of Grothendieck categories we obtain at once the *Categorical* or *Absolute Osofsky-Smith Theorem.*

Throughout this section $\mathcal{G}$ will denote a fixed *Grothendieck category,* and for any object $X \in \mathcal{G}$, $\mathcal{L}(X)$ will denote the upper continuous modular lattice of all subobjects of X. For any subobjects Y and Z of X we denote by $Y \cap Z$ their meet and by $Y + Z$ their join in the lattice $\mathcal{L}(X)$.

As we already defined in Section 3.2, if $\mathbb{P}$ is a property on lattices, an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$, and a subobject Y of X is/has $\mathbb{P}$ if the element Y of the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. As mentioned in Section 3.2, for a complement (respectively, compact) subobject of an object $X \in \mathcal{G}$ one uses the well established term of a *direct summand* (respectively, *finitely generated* subobject) of X; for this reason, instead of saying that X is a CC object we will say that X is a CS object (acronym for Closed subobjects are direct Summands).

Recall from Section 3.2 that the category $\mathcal{G}$ is called *locally finitely generated* if it has a family of finitely generated generators, or equivalently if the lattices $\mathcal{L}(X)$ are compactly generated for all objects X of $\mathcal{G}$. We say that an object $X \in \mathcal{G}$ is *locally finitely generated* if the lattice $\mathcal{L}(X)$ is compactly generated.

THEOREM 5.3.1. (CATEGORICAL O-ST). *Let $\mathcal{G}$ be a Grothendieck category, and let $X \in \mathcal{G}$ be a finitely generated, locally finitely generated object such that every finitely generated subfactor object of X is CS. Then X is a finite direct sum of uniform objects.*

PROOF. Apply Theorem 5.2.4 to the lattice $L = \mathcal{L}(X)$. □

An object X of a Grothendieck category $\mathcal{G}$ is called *CF* (acronym for Closed are Finitely generated) if every closed subobject of X is finitely generated, and *completely CF* if every quotient object of X is CF.

COROLLARY 5.3.2. *Let X be a finitely generated, locally finitely generated object of a Grothendieck category $\mathcal{G}$ such that every finitely generated subobject of X is completely CF. Then X is a finite direct sum of indecomposable subobjects.*

PROOF. Specialize Proposition 5.2.5 for the lattice $L = \mathcal{L}(X)$. □

Denote by $\mathcal{H}$ the class of all finitely generated objects of $\mathcal{G}$, and let $\mathcal{A}$ be a subclass of $\mathcal{H}$ satisfying the following three conditions:

(A_1) If $X \in \mathcal{A}$, $X' \in \mathcal{G}$, and $X \simeq X'$, then $X' \in \mathcal{A}$.
(A_2) If $X \in \mathcal{A}$ then $X/X' \in \mathcal{A}, \forall X' \subseteq X$.
(A_3) If $X \in \mathcal{A}$ and $Z \subseteq Y \subseteq X$ with $Y/Z \in \mathcal{A}$, then $\exists U \subseteq X$ such that $U \in \mathcal{A}$ and $Y = Z + U$.

As we have noticed in Section 3.2, the class $\mathcal{H}$ could be empty, and in this case everything that follows makes no sense.

THEOREM 5.3.3. (CATEGORICAL $\mathcal{A}$-O-ST). *Let $\mathcal{A}$ be a class of finitely generated objects of a Grothendieck category $\mathcal{G}$ satisfying the conditions $(A_1) - (A_3)$ above, and let $X \in \mathcal{A}$. Assume that all subfactors of X in $\mathcal{A}$ are CS. Then X is a finite direct sum of uniform objects of $\mathcal{G}$.*

PROOF. Specialize Theorem 5.2.6 for the lattice $L = \mathcal{L}(X)$. □

We present now a consequence, involving injective objects, of the Categorical O-ST. Recall that for any Grothendieck category one can define as in Mod-R the concepts of an *M-injective object, self-injective object, simple object,* and *semisimple object* (see, e.g., [**26**, p. 9]).

LEMMA 5.3.4. *Any self-injective object of a Grothendieck category $\mathcal{G}$ is a CS object.*

PROOF. See [**14**, Lemma 4.13]. □

PROPOSITION 5.3.5. *The following assertions are equivalent for a locally finitely generated object X of a Grothendieck category $\mathcal{G}$.*

(1) *X is semisimple.*
(2) *Every finitely generated subfactor of X is X-injective.*

Proof. (1) $\Longrightarrow$ (2) If X is semisimple then so is any subfactor of X, which is clearly X-injective.

(2) $\Longrightarrow$ (1) Let $V = Y/Z$, $Z \subseteq Y \subseteq X$, be a finitely generated subfactor of X. Then V is X-injective by hypothesis. It follows that V is X/Z-injective, and so, also Y/Z-injective by the well-known properties of X-injective objects (see, e.g., [26, Proposition 1.11]). Thus, V is self-injective, and consequently CS by Lemma 5.3.4.

Now let F be a finitely generated subobject of X. By Theorem 5.3.1, F is a finite direct sum of uniform objects. Let U be a uniform direct summand of F. Then, by hypothesis, any finitely generated subobject U' of U is X-injective, so it is a direct summand of X. Clearly U' is also a direct summand of the uniform object U. It follows that either $U' = 0$ or $U' = U$. Because X has been supposed to be locally finitely generated, for any $0 \neq W \subseteq U$, W is the sum of all its non-zero finitely generated subobjects, all of them being equal to U. Thus, U is a simple object of $\mathcal{G}$, and consequently F is a semisimple object of $\mathcal{G}$. Using again the fact that X is locally finitely generated, we conclude that X is a sum of simple objects, i.e., is semisimple. □

A nice result of Okado [77] states that a unital ring R is right Noetherian if and only if every CS right R-module can be expressed as a direct sum of indecomposable (or uniform) modules. We guess that the following categorical version of Okado's Theorem holds:

A Grothendieck category $\mathcal{G}$ is locally Noetherian if and only if every CS object of $\mathcal{G}$ can be expressed as a direct sum of indecomposable (*or uniform*) *objects.*

We end this section by mentioning that some statements/results of [79] and [45] related to the Categorical O-ST saying that "*basically the same proof for modules works in the categorical setting*" are not in order (see [14, p. 2670]). Such statements are very risky and may lead to incorrect results. One reason is that we cannot prove equality between two subobjects of an object in a category as we do for submodules by taking elements of them. Notice that the well-hidden errors in the statements/results occurring in the papers mentioned above on the Categorical O-ST could be spotted only by using our latticial approach of it. So, we do not only correctly absolutize the module-theoretical O-ST but also provide a correct proof of its categorical extension by passing first through its latticial counterpart.

5.4. The Relative Osofsky-Smith Theorem

In this section we present the relative version with respect to a hereditary torsion theory of the module-theoretical Osofsky-Smith Theorem [79, Theorem 1]. Its proofs is an easy application of the corresponding lattice-theoretical results of Sections 5.1 and 5.2.

Throughout this section R denotes a ring with non-zero identity, $\tau = (\mathcal{T}, \mathcal{F})$ a fixed hereditary torsion theory on Mod-R, and $\tau(M)$ the τ-torsion submodule of a right R-module M.

Recall from Section 3.4 some notation and terminology. For any module M_R we have denoted

$$\mathrm{Sat}_\tau(M) := \{\, N \mid N \leqslant M,\ M/N \in \mathcal{F}\,\},$$

which is an upper continuous modular lattice. If $\mathbb{P}$ is any property on lattices, we said that a module M is/has $\mathbb{P}$ if the lattice $\mathrm{Sat}_\tau(M)$ is/has $\mathbb{P}$, and a submodule N of M is/has $\mathbb{P}$ if its τ-saturation $\overline{N}$, which is an element of $\mathrm{Sat}_\tau(M)$, is/has $\mathbb{P}$. Thus, a module M_R is called *τ-CC* (or *τ-extending*) if the lattice $\mathrm{Sat}_\tau(M)$ is CC (or extending). However, in the sequel we shall use the more appropriate term of a *τ-CS* module (respectively, *τ-direct summand* of a module) instead of that of a *τ-CC* module (respectively, *τ-complement submodule* of a module).

Consider the quotient category $\text{Mod-}R/\mathfrak{T}$ of $\text{Mod-}R$ modulo its localizing subcategory $\mathfrak{T}$, and let

$$T_\tau : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathfrak{T}$$

be the canonical functor. We have seen in Proposition 4.2.3 that for any M_R, the mapping

$$\mathrm{Sat}_\tau(M) \longrightarrow \mathcal{L}(T_\tau(M)), \quad N \mapsto T_\tau(N),$$

is an isomorphism of lattices, so, for any property $\mathbb{P}$ on lattices, the module M_R is/has τ-$\mathbb{P}$ if and only if the object $T_\tau(M)$ in the quotient Grothendieck category $\text{Mod-}R/\mathfrak{T}$ is/has $\mathbb{P}$.

We present now intrinsic characterizations, that is, without explicitly referring to the lattice $\mathrm{Sat}_\tau(M)$, of the relative concepts that will appear in the Relative O-ST.

PROPOSITION 5.4.1. *The following assertions hold for a module M_R and $N \leqslant M$.*

(1) *N is τ-essential in $M \iff (\forall\, P \leqslant M,\, P \cap N \in \mathfrak{T} \implies P \in \mathfrak{T})$.*
(2) *M is τ-uniform $\iff (\forall\, P, K \leqslant M,\, P \cap K \in \mathfrak{T} \implies P \in \mathfrak{T}$ or $K \in \mathfrak{T})$.*
(3) *N is a τ-pseudo-complement in $M \iff \exists\, P \leqslant M$ such that $N \cap P \in \mathfrak{T}$ and N is maximal among the submodules of M having this property; in this case $N \in \mathrm{Sat}_\tau(M)$ and $N \cap \overline{P} = \tau(M)$.*
(4) *N is τ-closed in $M \iff \forall\, P \leqslant M$ such that $N \subseteq P$ and N is a τ-essential submodule of P one has $P/N \in \mathfrak{T}$. If additionally $N \in \mathrm{Sat}_\tau(M)$, then N is τ-closed in $M \iff N$ has no proper τ-essential extension in M.*
(5) *N is a τ-direct summand in $M \iff \exists\, P \leqslant M$ such that $M/(N+P) \in \mathfrak{T}$ & $N \cap P \in \mathfrak{T}$.*
(6) *M is τ-complemented $\iff \forall\, N \leqslant M, \exists\, P \leqslant M$ such that $M/(N+P) \in \mathfrak{T}$ & $N \cap P \in \mathfrak{T}$.*
(7) *M is τ-compact $\iff \forall\, N \leqslant M$ with $M/N \in \mathfrak{T}$, $\exists\, N' \leqslant N$ such that N' is finitely generated and $M/N' \in \mathfrak{T}$, in other words, the filter*
$$F(M) := \{\, N \leqslant M \mid M/N \in \mathfrak{T}\,\}$$
has a basis consisting of finitely generated submodules.
(8) *M is τ-CEK $\iff$ any τ-closed submodule of M is a τ-essential submodule of a τ-compact submodule of M.*
(9) *M is τ-compactly generated $\iff \forall\, N \leqslant M, \exists\, I_N$ a set and a family $(C_i)_{i\in I_N}$ of τ-compact submodules of M such that $\sum_{i\in I_N} C_i \subseteq N$ and $N/\big(\sum_{i\in I_N} C_i\big) \in \mathfrak{T}$.*

PROOF. Apply Lemmas 3.4.2, 3.4.3, and 3.4.4. Fore more details, see [**14**, Proposition 5.3]. □

We are now going to clarify the relations between the concepts of a τ-compact, τ-compactly generated, and τ-finitely generated module.

As in [**24**], a module M is said to be *τ-finitely generated* if there exists a finitely generated submodule M' of M such that $M/M' \in \mathfrak{T}$. Note that a τ-finitely generated module is not necessarily τ-compact. To see this, let R be an infinite direct product of copies of a field, let $\mathcal{A}$ be the localizing subcategory of Mod-R consisting of all semi-Artinian R-modules, and let τ_0 be the hereditary torsion theory on Mod-R defined by $\mathcal{A}$. We have seen at the end of Section 3.2 that the quotient category Mod-$R/\mathcal{A}$ has no simple object, so, in particular $\mathrm{Sat}_{\tau_0}(R_R)$ is not a compact lattice, i.e., R_R is not τ_0-compact, though R_R is a finitely generated R-module, in particular τ_0-finitely generated.

On the other hand, a τ-compact module M is necessarily τ-finitely generated: indeed, $M \in F(M)$ because $M/M \in \mathfrak{T}$, so, by Proposition 5.4.1(7), there exists a finitely generated submodule M' of M such that $M/M' \in \mathfrak{T}$, i.e., M is τ-finitely generated.

A τ-compactly generated module is not necessarily τ-compact: indeed, any module M_R which is not finitely generated is clearly ξ-compactly generated but not ξ-compact, where $\xi = (\{0\}, \mathrm{Mod}\text{-}R)$ is the trivial torsion theory on Mod-R. Conversely, we guess that a τ-compact module is not necessarily τ-compactly generated, but do not have any counterexample.

We say that a finite family $(N_i)_{1\leqslant i\leqslant n}$ of submodules of a module M_R is *τ-independent* if $N_i \notin \mathfrak{T}$ for all $1 \leqslant i \leqslant n$, and

$$N_{k+1} \cap \sum_{1\leqslant j\leqslant k} N_j \subseteq \tau(M),\ \forall\, k,\ 1 \leqslant k \leqslant n-1,$$

or, equivalently

$$\overline{N_{k+1}} \cap \overline{\sum_{1\leqslant j\leqslant k} N_j} = \overline{N_{k+1}} \wedge \Big(\bigvee_{1\leqslant j\leqslant k} \overline{N_j} \Big) = \tau(M),$$

in other words, by Lemma 2.2.1, the family $\big(\overline{N_i}\big)_{1\leqslant i\leqslant n}$ of elements of the lattice $\mathrm{Sat}_\tau(M)$ is independent. More generally, a family $(N_i)_{i\in I}$ of submodules of M is called *τ-independent* if the family $\big(\overline{N_i}\big)_{i\in I}$ of elements of the lattice $\mathrm{Sat}_\tau(M)$ is independent.

Theorem 5.4.2. (Relative O-ST). *Let M_R be a τ-compact, τ-compactly generated module. Assume that all τ-compact subfactors of M are τ-CS. Then there exists a finite τ-independent family $(U_i)_{1\leqslant i\leqslant n}$ of τ-uniform submodules of M such that $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathfrak{T}$.*

Proof. Let N/P, $P \leqslant N \leqslant M$, be a τ-compact subfactor of M. Observe that, in view of Lemma 3.4.4, the interval $[\overline{P}, \overline{N}]$ of $\mathrm{Sat}_\tau(M)$ is isomorphic to the compact lattice $\mathrm{Sat}_\tau(N/P)$, which, by hypothesis is CC. So, we can specialize the Latticial O-ST (Theorem 5.2.4) for the compact, compactly generated, modular lattice $L = \mathrm{Sat}_\tau(M)$ to deduce that there exists a finite independent family $(U_i)_{1\leqslant i\leqslant n}$ of uniform elements of L such that $M = \dot{\bigvee}_{1\leqslant i\leqslant n} U_i$ is the direct join in L of the family $(U_i)_{1\leqslant i\leqslant n}$. Thus, $(U_i)_{1\leqslant i\leqslant n}$ is a τ-independent family of τ-uniform submodules of M. Since

$$M = \dot{\bigvee}_{1\leqslant i\leqslant n} U_i = \overline{\sum_{1\leqslant i\leqslant n} U_i}\,,$$

it follows that $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathfrak{T}$, as desired. $\square$

We are now going to state a more simplified version of the Relative O-ST in case the given module M_R is τ-torsion-free. To do that, we need the following result.

LEMMA 5.4.3. *Let $M_R \in \mathcal{F}$ be a module, and let $(N_i)_{i\in I}$ be a family of submodules of M. Then, the following statements hold.*

(1) *$(N_i)_{i\in I}$ is an independent family of submodules of $M \iff \left(\overline{N_i}\right)_{i\in I}$ is an independent family of elements of the lattice $\mathrm{Sat}_\tau(M)$.*

(2) *M is τ-uniform $\iff$ M is uniform.*

PROOF. (1) The implication "$\Longleftarrow$" is clear. Conversely, let $(N_i)_{i\in I}$ be an independent family of submodules of M. In order to prove that $\left(\overline{N_i}\right)_{i\in I}$ is an independent family of elements of the lattice $\mathrm{Sat}_\tau(M)$, it is sufficient to assume that I is the finite set $\{1, \dots, n\}$ for some $n \in \mathbb{N}$, $n \geqslant 2$, because the independence is a property of finitary character in any upper continuous lattice, as $\mathrm{Sat}_\tau(M)$ is. If $\bigvee$ and $\bigwedge$ denote the join and meet, respectively, in the lattice $\mathrm{Sat}_\tau(M)$, then, for each $1 \leqslant k < n$, we have

$$\Big(\bigvee_{1\leqslant i\leqslant k} \overline{N_i}\Big) \bigwedge \overline{N_{k+1}} = \overline{\Big(\sum_{1\leqslant i\leqslant k} N_i\Big)} \bigcap \overline{N_{k+1}} = \overline{\Big(\sum_{1\leqslant i\leqslant k} N_i\Big) \bigcap N_{k+1}} = \overline{0} = \tau(M) = 0.$$

This proves, by Lemma 2.2.1, that $\left(\overline{N_i}\right)_{1\leqslant i\leqslant n}$ is an independent family of $\mathrm{Sat}_\tau(M)$, as desired.

(2) Observe that because $M \in \mathcal{F}$, for any $P \leqslant M$ we have $P \in \mathcal{T} \iff P = 0$; so, the result follows at once from Proposition 5.4.1(2). $\square$

THEOREM 5.4.4. (TORSION-FREE RELATIVE O-ST). *Let $M_R \in \mathcal{F}$ be a τ-compact, τ-compactly generated module. Assume that all τ-compact subfactors of M are τ-CS. Then, there exists a finite independent family $(U_i)_{1\leqslant i\leqslant n}$ of uniform submodules of M such that $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.*

PROOF. Use Lemma 5.4.3 in Theorem 5.4.2. $\square$

Since M is τ-$\mathbb{P}$ if and only if $M/\tau(M)$ is so, in view of Theorem 5.4.4 we can of course formulate the Relative O-ST in terms of essentiality and independence in the lattice $\mathcal{L}(M/\tau(M))$ instead of the relative ones in the lattice $\mathcal{L}(M)$:

THEOREM 5.4.5. *Let M_R be a τ-compact, τ-compactly generated module. If all τ-compact subfactors of M are τ-CS, then there exists a finite family $(U_i)_{1\leqslant i\leqslant n}$ of submodules of M, all containing $\tau(M)$, such that $(U_i/\tau(M))_{1\leqslant i\leqslant n}$ is an independent family of uniform submodules of $M/\tau(M)$ and $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.* $\square$

Recall that for a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on Mod-R we have denoted in Section 3.4 by

$$F_\tau := \{\, I \leqslant R_R \mid R/I \in \mathcal{T}\}$$

the *Gabriel filter* associated with τ. By a *basis* of the Gabriel filter F_τ we mean a subset B of F_τ such that every right ideal in F_τ contains some $J \in B$.

LEMMA 5.4.6. *The following statements hold for a hereditary torsion theory τ on* Mod-R.

(1) *The Gabriel filter F_τ has a basis consisting of finitely generated right ideals if and only if the right module R_R is τ-compact, i.e., the lattice $\mathrm{Sat}_\tau(R)$ is compact.*

(2) *If R is τ-Noetherian, then F_τ has a basis consisting of finitely generated right ideals of R.*

Proof. (1) is exactly Proposition 5.4.1(7) for $M = R_R$. For (2) see [**85**, Chapter XIII, Corollary 2.5]. $\square$

The converse in Lemma 5.4.6(2) is, in general, not true. Indeed, let $\xi = (\{0\}, \text{Mod-}R)$ be the trivial torsion theory on Mod-R for any non-Noetherian ring R. Then, the Gabriel filter $F_\xi = \{R\}$ has the basis $B = \{R\}$ with a single finitely generated ideal R, but R is not ξ-Noetherian.

Proposition 5.4.7. *The following assertions are equivalent for a Grothendieck category $\mathcal{G}$.*

(1) *$\mathcal{G}$ has a finitely generated generator.*
(2) *There exists a unital ring A and a hereditary torsion theory $\chi = (\mathcal{H}, \mathcal{E})$ on* Mod-A *such that the Gabriel filter F_χ has a basis of finitely generated right ideals of A and $\mathcal{G} \simeq \text{Mod-}A/\mathcal{H}$.*
(3) *There exists a unital ring A and a hereditary torsion theory $\chi = (\mathcal{H}, \mathcal{E})$ on* Mod-A *such that the lattice $\text{Sat}_\chi(A)$ is compact and $\mathcal{G} \simeq \text{Mod-}A/\mathcal{H}$.*

Proof. (2) $\iff$ (3) follows from Lemma 5.4.6(1).

(1) $\Longrightarrow$ (3) Let U be a finitely generated generator of $\mathcal{G}$, let R_U be the ring of endomorphisms of U, and let $S_U : \mathcal{G} \longrightarrow \text{Mod-}R_U$ be the functor $\text{Hom}_{\mathcal{G}}(U, -)$. Then S_U has a left adjoint $T_U : \text{Mod-}R_U \longrightarrow \mathcal{G}$. The Gabriel-Popescu Theorem says that T_U is an exact functor, $\text{Ker}(T_U) := \{\, M \in \text{Mod-}R_U \,|\, T_U(M) = 0 \,\}$ is a localizing subcategory of Mod-R_U, $T_U \circ S_U \simeq 1_{\mathcal{G}}$, and $\mathcal{G} \simeq \text{Mod-}R_U/\text{Ker}(T_U)$.

Set $A := R_U$, $\mathcal{H} := \text{Ker}(T_U)$, and $\chi := (\mathcal{H},\, \mathcal{E})$, where

$$\mathcal{E} = \{\, Y \in \mathcal{G} \,|\, \text{Hom}_{\mathcal{G}}(X, Y) = 0,\ \forall\, X \in \mathcal{H} \,\}.$$

Observe that $U \simeq (T_U \circ S_U)(U) \simeq T_U(A)$, so, by Proposition 4.2.3, the lattices $\mathcal{L}(U)$ and $\text{Sat}_\chi(A)$ are isomorphic; since $\mathcal{L}(U)$ is compact, so is also $\text{Sat}_\chi(A)$.

(3) $\Longrightarrow$ (1) Let H be the composition of the canonical functor $\text{Mod-}A \longrightarrow \text{Mod-}A/\mathcal{H}$ with the given equivalence functor $\text{Mod-}A/\mathcal{H} \longrightarrow \mathcal{G}$, and let $V := H(A_A)$. Then V is clearly a generator of $\mathcal{G}$. Since the canonical image of A_A in the quotient category $\text{Mod-}A/\mathcal{H}$ is finitely generated, so is V. $\square$

Theorem 5.4.8. *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on* Mod-R *such that its Gabriel filter F_τ has a basis consisting of finitely generated right ideals of R (in particular, this holds when R is τ-Noetherian), and let M_R be a τ-compact module. If all τ-compact subfactors of M are τ-CS, then there exists a finite family $(U_i)_{1\leqslant i\leqslant n}$ of submodules of M, all containing $\tau(M)$, such that $(U_i/\tau(M))_{1\leqslant i\leqslant n}$ is an independent family of uniform submodules of $M/\tau(M)$ and $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.*

Proof. Since F_τ has a basis consisting of finitely generated right ideals of R, the Grothendieck category $\text{Mod-}R/\mathcal{T}$ has a finitely generated generator by Proposition 5.4.7, so it is locally finitely generated. Thus, any module M_R is τ-compactly generated. Therefore, the result follows immediately from Theorem 5.4.5. $\square$

According to our definitions in Section 3.4 of module-theoretical concepts relative to a hereditary torsion theory τ, a module U_R is said to be *τ-simple* if the lattice $\mathrm{Sat}_\tau(U)$ is simple, which means that it has exactly two elements, i.e., $U \notin \mathcal{T}$ and $\mathrm{Sat}_\tau(U) = \{\tau(U), U\}$. Recall that U_R is called *τ-cocritical* if it is τ-simple and $U \in \mathcal{F}$.

The *τ-socle* of a module M_R, denoted by $\mathrm{Soc}_\tau(M)$, is defined as the τ-saturation of the sum of all τ-simple (or τ-cocritical) submodules of M, and M is said to be *τ-semisimple* if $M = \mathrm{Soc}_\tau(M)$. By [6, Proposition 1.15] or [18, Proposition 6.5(1)], $\mathrm{Soc}_\tau(M)$ is exactly the socle of the lattice $\mathrm{Sat}_\tau(M)$, and so, we have

$$M_R \text{ is a } \tau\text{-semisimple module} \iff \mathrm{Sat}_\tau(M) \text{ is a semi-atomic lattice} \iff$$
$$\iff T_\tau(M) \text{ is a semisimple object of the quotient category } \mathrm{Mod}\text{-}R/\mathcal{T}.$$

The next result is a relative version of the well-known *Osofsky's Theorem* [78].

Proposition 5.4.9. *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on* Mod-R *such that its Gabriel filter F_τ has a basis consisting of finitely generated right ideals of R (in particular, this holds when R is τ-Noetherian). Assume that R/I is an injective R-module for each $I \in \mathrm{Sat}_\tau(R)$. Then, any right R-module is τ-semisimple.*

Proof. See [14, Proposition 5.16]. $\square$

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