Hausdorff Nonstandard Extensions [*]

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ABSTRACT: We introduce the notion of Hausdorff extension of an arbitrary set X and we study the connections with the Stone-Čech compactification βX of the discrete space X. We characterize those Hausdorff extensions that satisfy the "transfer principle" of nonstandard analysis, and we investigate the consistency strength of their existence.

Contents

Introduction

Nonstandard analysis is often presented as a part of logic. This habit is, in our opinion, a mere historical accident. In fact, several different approches are possible. E.g., it is shown in [3] that every nonstandard model can be presented by means of a simple purely algebric construction. Other presentations using nonstandard set-theories have been started in [11] and [13]. Interesting and elementary approaches are also those of [12] and [2].

In this paper we exploit a topological approach to nonstandard models. In particular we construe the set of the hypernatural numbers as a topological extension of $\mathbb{N}$, where the "$*$" operator on subsets of $\mathbb{N}$ is nothing but the *closure* operator.

Let us define

- a Hausdorff space $(^*\mathbb{N}, \tau)$ is a *topological model* of the hypernatural numbers if the following conditions are satisfied

  (i) $\mathbb{N}$ is a dense discrete subspace of $^*\mathbb{N}$;

  (ii) every function $f : \mathbb{N} \to \mathbb{N}$ has a (unique) continuous extension $^*f : {^*\mathbb{N}} \to {^*\mathbb{N}}$;

  (iii) $\forall x \in \mathbb{N} \, . \, f(x) \neq g(x) \implies \forall \xi \in {^*\mathbb{N}} \, . \, {^*f}(\xi) \neq {^*g}(\xi)$;

  (iv) $\forall \xi, \eta \in {^*\mathbb{N}} \;\; \exists f, g \in \mathbb{N}^{\mathbb{N}} \;\; \exists \alpha \in {^*\mathbb{N}} \, . \, \xi = {^*f}(\alpha), \; \eta = {^*g}(\alpha)$.

In this paper we prove that a topological model of the hypernatural numbers is actually a nonstandard model of the natural numbers. In fact, it is possible to define a bounded elementary embeddding $* : V_\infty(\mathbb{N}) \to V_\infty(^*\mathbb{N})$ between the corresponding superstructures (see section 4.4 of [7] for the notation). In general, we refer to [10] for all the topological notions and facts used in this paper, and to [7] for definitions and facts concerning ultrapowers and ultrafilters.

Topological nonstandard extensions are studied in full generality in [9]. In this paper we restrict ourselves to analyse the case of Hausdorff spaces. This choice reduces dramatically the number of possibilities, and in fact it presents several interesting foundational questions and aspects. Namely, it turns out that the existence of a Hausdorff model of the hypernatural numbers is equivalent to the existence of an ultrafilter satisfying a property, labelled (C) in [8], which has been rarely considered in the literature. As far as we know, it is an open question whether the existence of ultrafilters of type (C) is provable in ZFC alone, while it follows easily from the Continuoum Hypothesis (see section 3 below).

On the other hand, all Haudorff topological models of the hypernatural numbers present some nice features. First of all they can be identified with suitable subspaces of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Thus any hypernatural number can be identified with an ultrafilter over $\mathbb{N}$. Moreover, once this identification is done, the function $^*f$ becomes the usual extension $\overline{f}$ of f to ultrafilters, namely $^*f(\alpha) = \overline{f}(\alpha) = \{f^{-1}(A) : A \in \alpha\}$.

Of particular interest is the case of topological models containing hypernatural numbers corresponding to *selective* ultrafilters. Given a Hausdorff extension $^*\mathbb{N}$ of $\mathbb{N}$ containing such an α, let us consider the subset $\mathcal{N} := \{^*f(\alpha) : f \in \mathbb{N}^\mathbb{N}\}$. Then $\mathcal{N}$ has the following remarkable properties:

- $\mathcal{N}$ is a nonstandard submodel of $^*\mathbb{N}$, and it is *minimal* in the sense that no proper subset of $\mathcal{N}$ is a proper elementary extension of $\mathbb{N}$;

- for every pair $\beta, \gamma \in \mathcal{N}$ there exists $f \in \mathbb{N}^\mathbb{N}$ such that $\beta = ^*f(\gamma)$, hence $\mathcal{N}$ is *simple*, i.e. it has no proper topological submodels;

- $\mathcal{N} = \{^*g(\alpha) \mid g : \mathbb{N} \to \mathbb{N} \text{ strictly increasing}\}$;

- $\mathcal{N}$ is a set of *numerosities* in the sense of [1];[1]

- the set of hyperreal numbers $\mathcal{R}$ obtained from $\mathcal{N}$ satisfies the *Strong Cauchy Principle* of [2], i.e. every positive infinitesimal $\varepsilon \in \mathcal{R}$ is equal to $^*f(\alpha)$ for suitable strictly decreasing function $f : \mathbb{N} \to \mathbb{R}$.

[1] We can give a semplified notion of *numerosity* as a surjective map $num : \mathcal{S} \to \mathcal{N}$, where $\mathcal{S} = \mathcal{P}(\bigcup_{n \in \mathbb{N}} \mathbb{N}^n)$, such that:
1. $num(A) = |A|$ if A is finite;
2. $num(A \cup B) = num(A) + num(B)$ if A and B are disjoint;
3. $num(A \times B) = num(A) \cdot num(B)$;
4. $num(A) < num(B) \iff num(A) = num(A')$ for some $A' \subsetneq B$.

The paper is organized as follows. In Section 1, we introduce the notion of Hausdorff extension of an arbitrary set X and we study its connections with the Stone-Čech compactification of the discrete space X. In Section 2, we consider elementary (nonstandard) Hausdorff extensions and we give suitable topological and algebraic characterizations. In Section 3 we deal with the consistency problem for Hausdorff extensions and other related questions.[2]

1. Hausdorff extensions

The main feature of nonstandard models of Analysis is the existence of a canonical extension $^{*}f : {}^{*}\mathbb{R} \to {}^{*}\mathbb{R}$ of any (standard) function $f : \mathbb{R} \to \mathbb{R}$. The similarity with the continuous extensions of functions in various compactifications of topological spaces suggests the following definition of *Hausdorff extension* of an arbitrary set X.

Definition 1.1 Let $^{*}X$ be a Hausdorff space, and let X be a *dense* subspace of $^{*}X$. Then $^{*}X$ is a *Hausdorff extension* of X if every function $f : X \to X$ has a *continuous extension* $^{*}f : {}^{*}X \to {}^{*}X$.

Notice that the continuous extension $^{*}f$ is uniquely determined, because X is dense in $^{*}X$, which is Hausdorff. It follows that all Hausdorff extensions satisfy several natural "preservation properties". In the following lemma we list some important and natural ones, concerning compositions, restrictions, ranges, identity, injective, surjective, and constant functions.

Lemma 1.1 *The following properties hold in any Hausdorff extension $^{*}X$ of X, for all functions $f, g : X \to X$:*

1. *$^{*}g \circ {}^{*}f = {}^{*}(g \circ f)$;*

2. *if f is the identity, then $^{*}f$ is the identity;*

3. *if $f(x) = g(x)$ for all $x \in A$, then $^{*}f(\xi) = {}^{*}g(\xi)$ for all $\xi \in \overline{A}$;*

4. *if f is 1-1 on $A \subseteq X$, then $^{*}f$ is 1-1 on $\overline{A}$; in particular $^{*}f$ is injective if and only if f is injective;*

5. *$^{*}f(\overline{A}) = \overline{f(A)}$ for all $A \subseteq X$; in particular $^{*}f$ is surjective if and only if f is surjective;*

6. *finite ranges are preserved, i.e. $^{*}f(^{*}X) = f(X)$ whenever $f(X)$ is finite.*

    Proof. Points $1, 2$, and 3 are immediate, by uniqueness, and 4 follows since a function is injective if and only if it has a left inverse.

    The inclusion $^{*}f(\overline{A}) \subseteq \overline{f(A)}$ holds for all continuous functions. Therefore 5 follows again from 1-3, because the restriction of $^{*}f$ to $\overline{A}$ has a right inverse.

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[2] The authors are grateful to A. Blass for useful information.

Finally, point 6 follows from 5, because $^*f(X) = \overline{f(X)} = f(X)$ whenever $f(X)$ is finite.

$\square$

Notice that a finite set X cannot have *proper* Haudorff extensions, because finite sets are closed in Haudorff spaces. Hence we may restrict ourselves to consider only infinite sets X. We intend to assign an *extension* $^*A \subseteq {}^*X$ to each *subset* $A \subseteq X$. This can be done in a natural way by considering the *characteristic function*

$$\chi_A(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{array} \right.$$

of any subset A of X. In the sequel we shall assume w.l.o.g. that $0, 1 \in X$, so as to extend all characteristic functions.

By point 6 of the above lemma, the extension of the characteristic function χ_A is the characteristic function χ_{*A} of a suitable subset $^*A$ of $^*X$, which turns out to be the closure $\overline{A}$ of A in $^*X$, by point 3 of the same lemma. Moreover the sets $^*A$ are a clopen basis of a 0-dimensional[4] topology.

Lemma 1.2 *Let $^*X$ be a Hausdorff extension of X. Then*

(i) *if $\chi_A : X \to X$ is the characteristic function of $A \subseteq X$, then the extension $^*\chi_A = \chi_{*A}$ is the characteristic function of $^*A = \overline{A}$, the closure of A in $^*X$.*

(ii) *$^*A$ is (cl)open for all $A \subseteq X$, and the closure map $* : A \mapsto {}^*A = \overline{A}$ is an isomorphism of the complete boolean algebra $\mathcal{P}(X)$ onto the field $\mathcal{S}(^*X)$ of all clopen subsets of $^*X$, whose inverse map is $C \mapsto C \cap X$.*

(iii) *For any $\xi \in {}^*X$, the set $\mathcal{U}_\xi = \{A \subseteq X \mid \xi \in {}^*A\}$ is an ultrafilter over X, which is mapped by $*$ onto a set of neighborhood of ξ in $^*X$. In particular $\mathcal{S}(^*X)$ is a clopen basis of a 0-dimensional topology not finer than that of $^*X$.*

Proof.   We have already seen that the extensions of characteristic functions are characteristic functions. Moreover $^*\chi_A(\overline{A}) \subseteq \{1\}$ and $^*\chi_A(\overline{X \setminus A}) \subseteq \{0\}$. Therefore $\overline{A} = {}^*A$ and $\overline{X \setminus A} = {}^*(X \setminus A)$ are a clopen partition of $^*X$.

The closure map commutes with complements of *clopen* sets and with binary unions. Moreover different clopen subsets of $^*X$ cannot have the same intersection with X, which is dense in $^*X$. Therefore the statement (ii) is completely proven.

Finally, for each $A \subseteq X$, $\{\overline{A}, \overline{X \setminus A}\}$ is a partition of $^*X$. Hence exactly one of A and $X \setminus A$ belongs to $\mathcal{U}_\xi$, which is therefore an ultrafilter. The remaining assertions of (iii) follow from (i) and (ii).

$\square$

When dealing with nonstandart models, the topology generated by $\mathcal{S}(^*X)$ is commonly called $\mathcal{S}$-topology. For sake of simplicity, we shaw directly assume in the sequel that the topology of $^*X$ is the $\mathcal{S}$-topology.

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[4] Recall that a space is 0-*dimensional* if the *clopen sets* are a *basis* of its topology.

It is apparent that the Stone-Čech compactification[5] βX of a discrete space X is a Hausdorff extension of X, since every function $f : X \to X$ has a unique continuous extension $\overline{f} : \beta X \to \beta X$.[6] In fact, it is *universal* in the sense that any Hausdorff extension of X is canonically homeomorphic to a suitable subspace of βX. More precisely:

Definition 1.2   Let $^*X$ be a Hausdorff extension of X. A subspace Y of $^*X$ is called **invariant* if $^*f(\eta) \in Y$ for all $\eta \in Y$ and all $f : X \to X$.

Theorem 1 *Every *invariant subspace of βX is a Hausdorff extension of X. Conversely, for every Hausdorff extension $^*X$ of X, there exists a unique continuous map $\jmath : {}^*X \to \beta X$ that extends the canonical embedding $e : X \to \beta X$, and $\jmath$ maps homeomorphically $^*X$ onto a *invariant subspace of βX.*

Precisely, for all $\xi \in {}^*X$, $\jmath(\xi)$ is the point of βX corresponding to the ultrafilter $\mathcal{U}_\xi$. Moreover, for all $f : X \to X$, $\jmath \circ {}^*f = \overline{f} \circ \jmath$.

Finally, $\jmath$ is surjective if and only if $^*X$ is compact. Hence βX is the unique compact Hausdorff extension of X.[7]

Proof. The first assertion is obviously the ground of the definition of *invariant subspace.

Assume that $\jmath$ maps any point $\xi \in {}^*X$ to the point of βX corresponding to the ultrafilter $\mathcal{U}_\xi$. Then, for all $x \in X$, $\mathcal{U}_x$ is the principal ultrafilter generated by x, and $\jmath$ induces the canonical embedding of X into βX. If $\mathcal{O}_A$ is a basic open set of βX, then $\jmath^{-1}(\mathcal{O}_A) = \overline{A}$, hence $\jmath$ is continuous, and so the unique continuous extension of e.

We have $\xi \in \overline{A} \iff {}^*f(\xi) \in \overline{f(A)}$ for all $\xi \in {}^*X$, or equivalently $A \in \mathcal{U}_\xi \iff f(A) \in \mathcal{U}_{*f(\xi)}$. It follows that $\overline{f}(\jmath(\xi)) = \jmath({}^*f(\xi))$.

The map $\jmath$ is injective, since $^*X$ is Hausdorff. Moreover $\mathcal{U}_\xi$ corresponds to the filter of the clopen neighborhoods of ξ, by Lemma 1.2.

Finally, $\jmath$ is surjective if and only if every ultrafilter over X is equal to $\mathcal{U}_\xi$ for suitable $\xi \in {}^*X$. This fact is equivalent to compactness, because every proper filter of closed subsets of $^*X$ has nonempty intersection if and only if every maximal filter in the field $\mathcal{S}({}^*X)$ has nonempty intersection.

□

From the above theorem we see that all Hausdorff extensions make a substantial use of the same "function-extending mechanism", namely that arising from the Stone-Čech compactification. We conclude this section by showing that any

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[5] For various definitions and properties of the Stone-Čech compactification see [10]. Here we only recall that if X is a discrete space, then βX can be identified with the set of all ultrafilters over X, endowed with the topology having the family $\{\mathcal{O}_A \mid A \in \mathcal{P}(X)\}$ as basis, where $\mathcal{O}_A$ is the set of all ultrafilters containing A. The embedding $e : X \to \beta X$ is given by the principal ultrafilters.

[6] In terms of ultrafilters, $\overline{f}$ can be defined by putting $A \in \overline{f}(\mathcal{U}) \Leftrightarrow f^{-1}(A) \in \mathcal{U}$.

[7] A detailed study of the canonical map $\jmath$ and its properties in the context of the $\mathcal{S}$-topology of arbitrary nonstandard models can be found in [14].

Hausdorff extension of X is the union of smaller *invariant subspaces, that can be naturally viewed as "quotients" of suitable ultrapowers of X.

In dealing with a ultrapower $X^X/\mathcal{U}$, where $\mathcal{U}$ is an arbitrary ultrafilter over X, we use the following notation:

- $[f] \in X^X/\mathcal{U}$ is the equivalence class of the function $f : X \to X$;

- $\overline{g} : X^X/\mathcal{U} \to X^X/\mathcal{U}$ is the interpretation of the function g in the ultrapower, i.e. $\overline{g}([f]) = [g \circ f]$ for all $f : X \to X$;

- $A^X/\mathcal{U} \subseteq X^X/\mathcal{U}$ is the interpretation of $A \subseteq X$ in the ultrapower.

The subsets $A^X/\mathcal{U}$, for $A \subseteq X$, are the basis of a 0-dimensional topology, which we call the *Star topology* of the ultrapower $X^X/\mathcal{U}$. We have

Lemma 1.3   *Let $^*X$ be a Hausdorff extension of X. For $\alpha \in {}^*X$ put*

$$^*X_\alpha = \{{}^*f(\alpha) \mid f : X \to X\}.$$

*Then $^*X_\alpha$ is the least *invariant subspace of X containing α.*

There exists a unique map $\psi : X^X/\mathcal{U}_\alpha \to {}^*X_\alpha$ such that $\quad ^*g \circ \psi = \psi \circ \overline{g} \quad$ for all $g : X \to X$. Moreover ψ is surjective, continuous and open w.r.t. the Star topology of $X^X/\mathcal{U}_\alpha$.

Proof.   The first assertion is immediate.

Define the map $\pi : X^X \to {}^*X_\alpha$ by $\pi(f) = {}^*f(\alpha)$. By Lemma 1.1 we have $\pi(f) = \pi(g)$ whenever $f = g$ on some $A \subseteq X$ s.t. $\alpha \in \overline{A}$, i.e. on some $A \in \mathcal{U}_\alpha$. Therefore π induces a map $\psi : X^X/\mathcal{U}_\alpha \to {}^*X_\alpha$, by putting $\psi([f]) = {}^*f(\alpha)$. We have, for all $f, g : X \to X$,

$$^*g(\psi([f])) = {}^*g({}^*f(\alpha)) = {}^*(g \circ f)(\alpha) = \psi([g \circ f]) = \psi(\overline{g}([f])).$$

Hence ψ is the unique map satisfying the wanted identity.

Finally, ψ is surjective by definition of $^*X_\alpha$, and for all $f \in X^X$

$$\psi([f]) \in {}^*A \iff \exists U \in \mathcal{U}_\alpha . f(U) \subseteq A \iff [f] \in A^X/\mathcal{U}_\alpha.$$

Therefore ψ maps elements of the basis of the Star topology of $X^X/\mathcal{U}_\alpha$ onto elements of the clopen basis of the topology of $^*X$.

$\square$

The above lemma suggests the following

Definition 1.3 Let $^*X$ be a Hausdorff extension of X. Then

- $^*X$ is *principal* if there exists $\alpha \in {}^*X$ such that

$$^*X = {}^*X_\alpha = \{{}^*f(\alpha) \mid f : X \to X\}.$$

- $^*X$ is *simple* if it has no proper nontrivial *invariant subspaces (equivalently $^*X = {}^*X_\alpha$ for all $\alpha \in {}^*X \setminus X$).

- $^*X$ is *coherent* if any two points of $^*X$ belong to some principal subspace $^*X_\alpha$ of $^*X$. (or equivalently for all $\xi, \eta \in {}^*X$ there exist functions $f, g : X \to X$ and $\alpha \in {}^*X$ s.t. $^*f(\alpha) = \xi$ and $^*g(\alpha) = \eta$).

Clearly every simple Hausdorff extension is principal. We can give the following complete characterization:

Theorem 2 *Let $^*X$ be a Hausdorff extension of X. Then the following properties are equivalent:*

(i) $^*X$ *is simple;*

(ii) $^*X$ *is coherent and the ultrafilter $\mathcal{U}_\alpha$ is selective[9] for all $\alpha \in {}^*X \setminus X$;*

(iii) *there exists $\alpha \in {}^*X$ such that $^*X = {}^*X_\alpha$ and $\mathcal{U}_\alpha$ is selective.*

If $^*X$ is simple, then the canonical map $\psi_\alpha : X^X/\mathcal{U}_\alpha \to {}^*X$ is bijective for any $\alpha \in {}^*X \setminus X$, hence it is a homeomorphism w.r.t. the Star topology. Moreover $^*X$ satisfies the following property, for all $f, g : X \to X$:

(∗) $f(x) \neq g(x)$ *for all $x \in X$* $\implies$ $^*f(\xi) \neq {}^*g(\xi)$ *for all $\xi \in {}^*X$.*

Conversely, every ultrapower of X modulo a selective ultrafilter over X, if endowed with the Star topology, becomes a simple Hausdorff extension of X satisfying the property (∗).

Proof.
$(i) \Rightarrow (ii)$. Assume that $^*X = {}^*X_\xi$ for all $\xi \in {}^*X \setminus X$. Then for all $\alpha \in {}^*X \setminus X \perp \xi = {}^*f(\alpha)$ and $\alpha = {}^*g(\xi) = {}^*g({}^*f(\alpha))$, for suitable $f, g : X \to X$. Hence $\mathcal{U}_\alpha = \overline{g \circ f}(\mathcal{U}_\alpha)$. This implies that $[g \circ f]$ is the class of the identity, and so f is equivalent to a bijective function modulo $\mathcal{U}_\alpha$. Therefore all ultrafilters $\mathcal{U}_\alpha$ are selective.

$(ii) \Rightarrow (iii)$. It is enough to show that $^*X$ is principal. Let $\alpha, \beta \in {}^*X \setminus X$ and pick $\xi \in {}^*X$ such that $\alpha = {}^*f(\xi)$ and $\beta = {}^*g(\xi)$ for suitable functions f, g. Since $\mathcal{U}_\xi$ is selective, both f and g can be taken bijective. Then $\beta = {}^*(g \circ f^{-1})(\alpha)$, and so α is a generator.

$(iii) \Rightarrow (i)$. Assume that $^*X = {}^*X_\alpha$, with $\mathcal{U}_\alpha$ selective. Given $\xi \in {}^*X \setminus X$ pick $f : X \to X$ such that $\xi = {}^*f(\alpha)$. Then f is not equivalent to a constant, hence it is equivalent to a bijective function g. Now $^*g(\alpha) = {}^*f(\alpha) = \xi$, and $\alpha = {}^*g^{-1}(\xi)$. It follows that $^*X$ is simple, since $^*X_\alpha = {}^*X_\xi$ for all $\xi \in {}^*X \setminus X$.

If the ultrafilter $\mathcal{U}_\alpha$ is selective and $^*f(\alpha) = {}^*g(\alpha)$, then f, g can be assumed bijective. Moreover $g \circ f^{-1}$ is the identity modulo $\mathcal{U}_\alpha$, hence $[f] = [g]$. Therefore ψ_α

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[9] Many equivalent properties can be used in defining selective ultrafilters (see, e.g. [7] or [5]). Here we need the following: *$\mathcal{U}$ is selective if and only if every $f : X \to X$ is either equivalent to a constant or to a bijective function.*

is injective, and actually a homeomorphism, being continuous and open. Moreover, if $^*f(\xi) = {}^*g(\xi)$ for $\xi = {}^*h(\alpha)$, say, then $[f \circ h] = [g \circ h]$, and so f, g agree on $h(U)$ for some $U \in \mathcal{U}_\alpha$. Therefore $^*X_\alpha$ satisfies the property $(*)$.

The last assertion of the theorem is a straightforward consequence of the preceding arguments.

$\square$

We shall see in the next section that the canonical map ψ_α is bijective if and only if $^*X_\alpha$ satisfies the property $(*)$. This fact is the corner stone in using Hausdorff extensions as *nonstandard models*. Therefore we conclude this section by isolating these extensions in the following

Definition 1.4 A Hausdorff extension $^*X$ of X is a *Hausdorff *extension* if, for all $f, g : X \to X$,

$(*)$ $f(x) \neq g(x)$ for all $x \in X$ $\implies$ $^*f(\xi) \neq {}^*g(\xi)$ for all $\xi \in {}^*X$.

2. Hausdorff elementary extensions

The main tool of the so called *nonstandard methods* is the study of extensions which preserve those properties of the standard structure which are relevant in the given context. The *Transfer* (*Leibniz*) *Principle* states that all properties that are expressible in a(*n in*)sufficiently expressive language are preserved by passing to the nonstandard models.

A crucial property of any nonstandard model of Analysis is the following:

(e) $^*f(\xi) = {}^*g(\xi) \iff \exists A \subseteq X \; (\xi \in {}^*A \; \& \; \forall x \in A \, . \, f(x) = g(x)),$

which expresses *preservation of equalizers*, i.e.

$$\{\xi \in {}^*X \mid {}^*f(\xi) = {}^*g(\xi)\} = {}^*\{x \in X \mid f(x) = g(x)\}.$$

This property has an "analytic" flavour, and in fact it is the very characteristic feature of *nonstandard* extensions when compared with *continuous* extensions of functions.

Since the inclusion $\overline{\{x \in X \mid f(x) = g(x)\}} \subseteq \{\xi \in {}^*X \mid {}^*f(\xi) = {}^*g(\xi)\}$ is obviously true in all Hausdorff extensions (point 3 of Lemma 1.1), any hypothesis yielding the rightpointed arrow suffices to obtain (e).

This turns out to be the case of property $(*)$, which states that *disjoint functions have disjoint extensions*, and thus corresponds to the particular case of *empty equalizers*. To be sure, the very ground of our use of $(*)$ in characterizing *Hausdorff *extensions* lies in the fact that this apparently weaker assumption yields indeed the whole of (e).

Lemma 2.1 *A Hausdorff extension $^*X$ of X satisfies the property* (e) *if and only if it is a Hausdorff *extension.*

Proof Since $(*)$ is a particular case of (e), we have only to prove that the former implies the inclusion

$$\{\xi \in {}^*X \mid {}^*f(\xi) = {}^*g(\xi)\} \subseteq \overline{\{x \in X \mid f(x) = g(x)\}}.$$

In fact, put $A = \{x \in X \mid f(x) = g(x)\}$, and let the functions f', g' agree with f, g outside A, and with the constants $0, 1$ on A, respectively. Since f', g' are disjoint on X, also ${}^*f'$ and ${}^*g'$ are disjoint on ${}^*X$, by $(*)$. But f, f' and g, g' agree on $X \setminus A$, hence ${}^*f' = {}^*f$ and ${}^*g' = {}^*g$ outside $\overline{A}$, by Lemma 1.1. Therefore $\{\xi \in {}^*X \mid {}^*f(\xi) = {}^*g(\xi)\} \subseteq \overline{A}$.

$\square$

We are now able to characterize all Hausdorff *extensions of X:

Theorem 3 *The following properties are equivalent for any principal Hausdorff extension ${}^*X_\alpha$ of X:*

1. *${}^*X_\alpha$ is a Hausdorff *extension of X;*

2. *the map $\psi : X^X/\mathcal{U}_\alpha \to {}^*X_\alpha$ is bijective;*

3. *the ultrafilter $\mathcal{U}_\alpha$ satisfies the property*

(C) $\forall f, g \in X^X \, (\, \overline{f}(\mathcal{U}_\alpha) = \overline{g}(\mathcal{U}_\alpha) \iff \exists A \in \mathcal{U}_\alpha \, \forall x \in A \, f(x) = g(x) \,).$[11]

*More generally, a Hausdorff extension ${}^*X$ of X is a Hausdorff *extension if and only if all ultrafilters $\mathcal{U}_\xi$, for $\xi \in {}^*X$, have the property (C), or equivalently if and only if all principal subextensions ${}^*X_\xi$ of ${}^*X$ are principal Hausdorff *extension.*

Proof. The left member of (C) is equivalent to ${}^*f(\alpha) = {}^*g(\alpha)$, whereas the right member is equivalent to $[f] = [g]$. Therefore each of the assertions 1-3 is equivalent to the condition (e), which characterizes *extensions.

The same argument proves also the last part of the lemma.

$\square$

It should be clear that the map $\psi : X^X/\mathcal{U}_\alpha \to {}^*X_\alpha$ is an isomorphism of first-order structures in the language having X as a set of constants and X^X as a set of function symbols. Therefore any Hausdorff *extension provides plenty of nonstandard models in a plain, natural way. Moreover these models are *strong extensions*, in the sense that they can be uniformly expanded to arbitrary first order languages.

That also n-ary relations and functions can be naturally extended is suggested by the following

Lemma 2.2 *Let ${}^*X$ be a Hausdorff *extension of X and let $\xi, \eta \in {}^*X$ satisfy ${}^*f_i(\xi) = {}^*g_i(\eta)$, for suitable functions $f_1, \ldots, f_n, g_1, \ldots, g_n \in X^X$. Then, for all $\varphi : X^n \to X$,*

$$ {}^*(\varphi \circ (p_1, \ldots, p_n))(\xi) = {}^*(\varphi \circ (q_1, \ldots, q_n))(\eta), $$

provided that there are $p, q \in X^X$ and $\alpha \in {}^*X$ with ${}^*p(\alpha) = \xi$ and ${}^*q(\alpha) = \eta$.

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[11] The property (C) has been labelled so in [8], where various connected properties of ultrafilters are considered.

Proof. Put $p_i = f_i \circ p$ and $q_i = g_i \circ q$: then $^*p_i(\alpha) = {}^*q_i(\alpha)$. Applying (e) n times, one obtains a set $A \subseteq X$ s.t. $\alpha \in \overline{A}$ and, for all $a \in A$, $p_1(a) = q_1(a), \ldots, p_n(a) = q_n(a)$. Hence $(\varphi \circ (p_1, \ldots, p_n))(a) = (\varphi \circ (q_1, \ldots, q_n))(a)$, and applying (e) again $^*(\varphi \circ (p_1, \ldots, p_n))(\alpha) = {}^*(\varphi \circ (q_1, \ldots, q_n))(\alpha)$.

Now we have $^*(\varphi \circ (f_1, \ldots, f_n))(\xi) = {}^*(\varphi \circ (f_1, \ldots, f_n))(^*p(\alpha)) = {}^*((\varphi \circ (f_1, \ldots, f_n)) \circ p)(\alpha) = {}^*(\varphi \circ (p_1, \ldots, p_n))(\alpha)$, and similarly $^*(\varphi \circ (g_1, \ldots, g_n))(\eta) = {}^*(\varphi \circ (q_1, \ldots, q_n))(\alpha)$.

$\square$

Let $^*X$ be a Hausdorff *extension of X. Given any first-order structure $\mathfrak{X} = \langle X; R, \ldots \rangle$ with universe X, and relations $R, \ldots$, we define a corresponding structure $^*\mathfrak{X} = \langle ^*X; {}^*R, \ldots \rangle$ with universe $^*X$ by putting

$$^*R = \{(^*f_1(\alpha), \ldots, {}^*f_n(\alpha)) \mid \alpha \in {}^*X, f_i \in X^X \; {}^*(\chi_R \circ (f_1, \ldots, f_n))(\alpha) = 1\},$$

where χ_R is the characteristic function of the n-ary relation R.

The above lemma clarifies that this definition is a "good" one (at least for *coherent* Hausdorff *extensions), and we can so give a complete characterization of the Hausdorff extensions which are elementary extensions.

Theorem 4 *Let $^*X$ be a Hausdorff extension of X, and let $\mathfrak{X}$ be a first order structure with universe X. Then the corresponding structure $^*\mathfrak{X}$ is a strong elementary extension of $\mathfrak{X}$ if and only if $^*X$ is a coherent Hausdorff *extension.*

*In particular, if $^*X$ is a principal Hausdorff *extension, then the structure $^*\mathfrak{X}$ is an ultrapower extension of $\mathfrak{X}$.*

Proof. We have already seen that the property $(*)$ is an instance of the transfer principle, and so it has to hold in any elementary extension. Although coherence may appear to be a completely different kind of property, nevertheless it turns out that a *strong uniform version* of coherence can be obtained by transfer as well, namely

(p) *There exist $p_1, p_2 : X \to X$ such that for all $\xi, \eta \in {}^*X$ there is a unique $\zeta \in {}^*X$ satisfying $\xi = {}^*p_1(\zeta)$, $\eta = {}^*p_2(\zeta)$.*

In fact, since X is infinite, there is a bijective map $\delta : X \to X \times X$ (*encoding of pairs*). Let $\pi_1, \pi_2 : X \times X \to X$ be the ordinary projections, and put $p_i = \pi_i \circ \psi$: then clearly for all $x, y \in X$ there is a unique $z \in X$ such that $x = p_1(z)$, $y = p_2(z)$. Thus a direct application of transfer yields (p), which holds therefore in all elementary extensions.

It remains to prove that any coherent Hausdorff *extension $^*X$ becomes a strong elementary extension through the above interpretations of the relation symbols. Now coherence makes $^*X$ the directed union of its principal subspaces $^*X_\alpha$, and Lemma 2.2 grants that every $^*X_\alpha$ is expanded so as to remain isomorphic to the corresponding ultrapower $X^X/\mathcal{U}_\alpha$. Therefore $^*X$, being the directed union of strong elementary extensions, is itself a strong elementary extension (see [7]).

$\square$

Differently from selectiveness, the property (C) seems *prima facie* rather weak, since it is always verified whenever any of the involved functions is injective. Nevertheless it turns out that even the consistency strength of the mere existence of Hausdorff *extensions is not yet exactly measured, as we shall see in the next section.

3. Final remarks

As shown by Theorems 2 and 3, Hausdorff extensions are strictly related to special ultrafilters, namely those having the so-called "property (C)", and in particular to selective ultrafilters. Unfortunately, while selectiveness has been deeply investigated, not much is known about the properties and the foundational strength of (C)-ultrafilters. What is known from the literature is essentially the fact that, over a countable set, the property (C) follows from the 3-arrow property of [4], which in turn is satisfied both by selective ultrafilters and by products of pairs of nonisomorphic selective ultrafilters.

We itemize below a few known facts that are relevant in our context.

- There are no countably incomplete selective uniform ultrafilters over an uncountable set X (see e.g. [7]).

- If the Continuum Hypothesis CH (or Martin Axiom MA)[13] holds, then there exist $2^{2^{\aleph_0}}$ selective ultrafilters over $\mathbb{N}$, and also $2^{2^{\aleph_0}}$ non-selective ultrafilters over $\mathbb{N}$ having the property (C)(see e.g. [7,4]).

- There are both models of ZFC with no selective ultrafilters, and models of ZFC with exactly one selective ultrafilter (up to isomorphisms) (see [15]).

As a straight consequence of the above, we have the following facts:

1. It is consistent with ZFC that there exist *both simple and non-simple* Hausdorff *extensions.

2. It is consistent with ZFC that there exist *no simple* Hausdorff extensions.

3. It is consistent with ZFC that there exists *a unique* simple Hausdorff extension (which is necessarily a *extension).

The latter property seems of particular interest to us, because it allows for the existence of a "*canonical*" (unique minimal) nonstandard model $^*X$ for any given X. However, as far as we do not abide ZFC as our foundational theory, we are not able to prove the existence of Hausdorff *extensions. This unpleasant fact led the authors to investigate also non-Hausdorff topological extensions: we refer to [9] for an extensive treatment of topological extensions in a more general setting.

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[13] Recall that the axiom MA is independent of ZFC and strictly weaker than CH.

References

1. V. Benci, M. Di Nasso - Numerosities of labelled sets: a new way of counting, *Adv. Math.*, 173 (2003),50-67.

2. V. Benci, M. Di Nasso - Alpha-Theory: an elementary axiomatics for nonstandard analysis,Expo. Math(to apear).

3. V. Benci, M. Di Nasso - A purely algebraic characterization of the hyperreal numbers, Proc. Amer. Math. Soc. (to apear).

4. J.E. Baumgartner, A.D. Taylor - Partitions theorems and ultrafilters, *Trans. Amer. Math. Soc.*, **241** (1978), 283–309.

5. A. Blass - A model-theoretic view of some special ultrafilters, in *Logic Colloquium '77* (A. MacIntyre, L. Pacholski, J. Paris, eds.), North Holland, Amsterdam 1978, 79–90. *J. Symb. Logic*, **46** (1981), 147–157.

6. D. Booth - Ultrafilters on a countable set, *Ann. Math. Logic*, **2** (1970/71), 1–24.

7. C.C. Chang, H.J. Keisler - *Model Theory* (3rd edition), North-Holland, Amsterdam 1990.

8. M. Daguenet-Teissier - Ultrafiltres à la façon de Ramsey, *Trans. Amer. Math. Soc.*, **250** (1979), 91–120.

9. M. Di Nasso, M. Forti - *Topological and nonstandard extensions*,Submited.

10. R. Engelking - *General Topology*, Polish Sci. Pub., Warszawa 1977.

11. K. Hrbàček - Axiomatic foundations for Nonstandard Analysis, *Fund. Math.* **98** (1978), 1–19, abstract in J.Syurb.Logic 41(1976),285.

12. H.J. Keisler - *Foundations of Infinitesimal Calculus*, Prindle, Weber and Schmidt, Boston 1976.

13. E. Nelson - Internal Set Theory; a new approach to nonstandard analysis, *Bull. Amer. Math. Soc.* **83** (1977), 1165–1198.

14. S. Ng, H. Render - The Puritz order and its relationship to the Rudin-Keisler order, in *Reuniting the Antipodes - Constructive and Nonstandard Views of the Continuum* (P. Schuster, U. Berger and H. Osswald, eds.), *Synthèse Library* **306**, Kluwer A. P., 2001, 157–166.

15. S. Shelah - *Proper Forcing*, Springer L. N. in Math. **940**, Berlin 1982.

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