Some generalizations in certain classes of rings with involution

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ABSTRACT: Let $R$ be a 2-torsion free $\sigma$-prime ring with an involution $\sigma$, $I$ a nonzero $\sigma$-ideal of $R$. In this paper we explore the commutativity of $R$ satisfying any one of the properties: (i) $d(x) \circ F(y) = 0$ for all $x, y \in I$. (ii) $[d(x), F(y)] = 0$ for all $x, y \in I$. (iii) $d(x) \circ F(y) = x \circ y$ for all $x, y \in I$. (iv) $d(x)F(y) - xy \in Z(R)$ for all $x, y \in I$. We also discuss $(\alpha, \beta)$-derivations of $\sigma$-prime rings and prove that if $G$ is an $(\alpha, \beta)$-derivation which acts as a homomorphism or as an anti-homomorphism on $I$, then $G = 0$ or $G = \beta$ on $I$.

Key Words: $\sigma$-prime ring; derivation; generalized derivation; $(\alpha, \beta)$-derivation; commutativity.

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1. Introduction

Throughout the present paper $R$ will denote an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes the anti-commutator $xy + yx$. In all that follows the symbol $Sa_\sigma(R)$, first introduced by Oukhtite, will denote the set of symmetric and skew symmetric elements of $R$, i.e. $Sa_\sigma(R) = \{ x \in R \mid \sigma(x) = \pm x \}$. An involution $\sigma$ of a ring $R$ is an anti-automorphism of order 2 (i.e. an additive mapping satisfying $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2(x) = x$ for all $x, y \in R$.) An ideal $I$ of $R$ is said to be a $\sigma$-ideal if $\sigma(I) = I$. An example, due to Rehman: Let $Z$ be the ring of integers. Set $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Z \right\}$. We define a map $\sigma : R \to R$ as follows:

$\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$. It is easy to check that $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in Z \right\}$ is a $\sigma$-ideal of $R$. Note that an ideal $I$ of a ring $R$ may be not a $\sigma$-ideal: Let $Z$ be the ring of integers and let $R = Z \times Z$. Consider a map $\sigma : R \to R$ defined by $\sigma((a, b)) = (b, a)$ for all $(a, b) \in R$. For an ideal $I = Z \times \{0\}$ of $R$, $I$ is not a $\sigma$-ideal.
of $R$ since $\sigma(I) = \{0\} \times \mathbb{Z} \neq I$. A ring $R$ is called 2-torsion free, if whenever $2x = 0$, with $x \in R$, then $x = 0$. Recall that a ring $R$ is prime if for any $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$. A ring $R$ equipped with an involution $\sigma$ is said to be a $\sigma$-prime ring if for any $a, b \in R$, $aRb = aR\sigma(b) = 0$ implies $a = 0$ or $b = 0$. It is worthwhile to note that every prime ring having an involution $\sigma$ is $\sigma$-prime but the converse is in general not true. Such an example due to Oukhtite is as following: Let $R$ be a prime ring, $S = R \times R^e$ where $R^e$ is the opposite ring of $R$, define $\sigma(x, y) = (y, x)$. From $(0, x)S(x, 0) = 0$, it follows that $S$ is not prime. For the $\sigma$-primeness of $S$, we suppose that $(a, b)S(x, y) = 0$ and $(a, b)S\sigma((x, y)) = 0$, then we get $aRx \times yRb = 0$ and $aRy \times xRb = 0$, and hence $aRx = yRb = aRy = xRb = 0$, or equivalently $(a, b) = 0$ or $(x, y) = 0$. This example shows that every prime ring can be injected in a $\sigma$-prime ring and from this point of view $\sigma$-prime rings constitute a more general class of prime rings. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \rightarrow R$ is called a generalized derivation associated with $d$ if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Let $\alpha$ and $\beta$ be homomorphisms of $R$, an additive mapping $G : R \rightarrow R$ is called an $(\alpha, \beta)$-derivation if $G(xy) = G(x)\alpha(y) + \beta(x)G(y)$ holds for all $x, y \in R$. Obviously, every $(1, 1)$-derivation on $R$ is just a derivation on $R$, where 1 is the identity mapping. Let $S$ be a nonempty subset of $R$ and $G$ an $(\alpha, \beta)$-derivation of $R$. If $G(xy) = G(x)G(y)$ or $G(xy) = G(y)G(x)$ for all $x, y \in S$, then $G$ is called an $(\alpha, \beta)$-derivation which acts as a homomorphism or anti-homomorphism on $S$.

Recently, some well-known results concerning prime rings have been proved for $\sigma$-prime rings by Oukhtite et al. (see [1-9], where further references can be found). Over the past thirty years, there has been an ongoing interest concerning the relationship between the commutativity of a prime ring $R$ and the behavior of a special mapping on that ring ([13], where further references can be found). In the year 2005, Ashraf et al. [10] proved some commutativity theorems for prime rings. In Section 3, we will generalize these results to generalized derivations on rings with involution.

On the other hand, Bell and Kappe [11] proved that if $d$ is a derivation of a prime ring $R$ which acts as a homomorphism or an anti-homomorphism on a nonzero ideal $I$ of $R$, then $d = 0$ on $R$. In [12], Albas and Argac extended this result to generalized derivations. Further, Oukhtite [8] proved the above result is also true for $\sigma$-prime rings. In Section 4, we extend the mentioned result in the setting of $(\alpha, \beta)$-derivations of $\sigma$-prime rings.

2. Some preliminaries

In all that follows, we assume that $R$ is a 2-torsion free $\sigma$-prime ring, where $\sigma$ is an involution of $R$. We begin with the following results which will be used to prove our theorems.
Lemma 2.1 (1, Lemma 3.1). Let $R$ be a 2-torsion free $\sigma$-prime ring and $I$ a nonzero $\sigma$-ideal of $R$. If $a, b \in R$ such that $aIb = aI\sigma(b) = 0$, then $a = 0$ or $b = 0$.

Lemma 2.2 (2, Lemma 2.3). Let $R$ be a 2-torsion free $\sigma$-prime ring, $I$ a nonzero $\sigma$-ideal and $d$ a derivation on $R$ commuting with $\sigma$. If $d^2(I) = 0$, then $d = 0$.

Lemma 2.3 (1, Theorem 3.2). Let $R$ be a 2-torsion free $\sigma$-prime ring, $I$ a nonzero derivation and $I$ a nonzero $\sigma$-ideal of $R$. If $d(I) \subseteq Z(R)$, then $R$ is commutative.

Lemma 2.4 (2, Theorem 1.2). Let $R$ be a 2-torsion free $\sigma$-prime ring, $I$ a nonzero $\sigma$-ideal and $d$ a nonzero derivation on $R$ commuting with $\sigma$. If $[d(x), d(y)] = 0$ for all $x, y \in I$, then $R$ is commutative.

3. Generalized derivations of $\sigma$-prime rings

Theorem 3.1 Let $R$ be a 2-torsion free $\sigma$-prime ring with an involution $\sigma$, $I$ a nonzero $\sigma$-ideal. If $R$ admits a nonzero generalized derivation $F$ associated a nonzero derivation $d$ commuting with $\sigma$ such that $d(x) \circ F(y) = 0$ for all $x, y \in I$, then $R$ is commutative.

Proof: By hypothesis, we have $d(x) \circ F(y) = 0$ for all $x, y \in I$. Replacing $y$ by $yr$ to get $d(x) \circ F(yr) = 0$, which implies that

$$(d(x) \circ y)d(r) - y[d(x), d(r)] + (d(x) \circ F(y))r - F(y)[d(x), r] = 0$$

for all $x, y \in I$ and $r \in R$. Now using that $d(x) \circ F(y) = 0$, the relation (1) yields that

$$(d(x) \circ y)d(r) - y[d(x), d(r)] - F(y)[d(x), r] = 0$$

which can reduce to

$$(d(x) \circ y)d^2(x) - y[d(x), d^2(x)] = 0$$

if we replace $r$ by $d(x)$, for all $x, y \in I$ and $r \in R$. Replacing $y$ by $zy$ in (2) to get

$$(d(x) \circ zy)d^2(x) - zy[d(x), d^2(x)] = 0$$

which implies that

$$z(d(x) \circ y)d^2(x) + [d(x), z]yd^2(x) - zy[d(x), d^2(x)] = 0$$

for all $x, y, z \in I$. In view of (2), the above relation leads to the following

$$[d(x), y]zd^2(x) = 0$$

for all $x, y, z \in I$.

Since $I$ is a $\sigma$-ideal and $d\sigma = \sigma d$, for all $x \in I \cap S\sigma(R)$, we have either $[d(x), y] = 0$ or $d^2(x) = 0$ by Lemma 2.1. Using the fact that $x - \sigma(x) \in I \cap S\sigma(R)$ for all $x \in I$, then $[d(x - \sigma(x)), y] = 0$ or $d^2(x - \sigma(x)) = 0$ for all $y \in I$.

If $[d(x - \sigma(x)), y] = 0$, then $[d(x), y] = [\sigma(d(x)), y]$, for all $y \in I$. As $I$ is a $\sigma$-ideal, it follows from (3) that $[d(x), y]zd^2(x) = 0 = \sigma([d(x), y)]zd^2(x)$, and hence Lemma 2.1 yields that $[d(x), y] = 0$ or $d^2(x) = 0$.  

If \( d^2(x - \sigma(x)) = 0 \), then \( d^2(x) = \sigma(d^2(x)) \) and (6) gives \([d(x), y] = 0 \) or \( d^2(x) = 0 \). Consequently, for all \( x \in I \), either \([d(x), I] = 0 \) or \( d^2(x) = 0 \).

Now let \( I_1 = \{ x \in I \mid [d(x), I] = 0 \} \) and \( I_2 = \{ x \in I \mid d^2(x) = 0 \} \). Then \( I_1, I_2 \) are both additive subgroups of \( I \) and \( I_1 \cup I_2 = I \). But a group can’t be a union of its two proper subgroups, and hence \( I_1 = I \) or \( I_2 = I \). On the one hand, if \( I_1 = I \), then

\[
[d(x), y] = 0, \tag{4}
\]

for all \( x, y \in I \). Replacing \( y \) by \( ry \) in (4) to get \([d(x), r]y = 0 \) for all \( x, y \in I \) and \( r \in R \). As \( d \) commutes with \( \sigma \), the fact that \( I \) is a \( \sigma \)-ideal gives us \([d(x), r] = 0 \) i.e. \( d(I) \subseteq Z(R) \), and hence \( R \) is commutative by Lemma 2.3. Of course, we can also replace \( y \) by \( yd(z) \) in (4) and use (4) to get \([yd(x), d(z)] = 0 \) for all \( x, y, z \in I \). As \( d \) commutes with \( \sigma \), the fact that \( I \) is a \( \sigma \)-ideal shows that \([d(x), d(z)] = 0 \) for all \( x, z \in I \), and hence \( R \) is commutative by Lemma 2.4. On the other hand, if \( I_2 = I \), then \( d^2(x) = 0 \) for all \( x \in I \). In other words, \( d^2(I) = 0 \) and hence \( d = 0 \) by Lemma 2.2, a contradiction. \( \square \)

**Theorem 3.2** Let \( R \) be a 2-torsion free \( \sigma \)-prime ring with an involution \( \sigma \), \( I \) a nonzero \( \sigma \)-ideal. If \( R \) admits a nonzero generalized derivation \( F \) associated a nonzero derivation \( d \) commuting with \( \sigma \) such that \([d(x), F(y)] = 0 \) for all \( x, y \in I \), then \( R \) is commutative.

**Proof:** We are given that

\[
[d(x), F(y)] = 0 \tag{5}
\]

for all \( x, y \in I \). Replacing \( y \) by \( yz \) in (5) and using (5) to get

\[
F(y)[d(x), z] + [y, d(x), d(z)] + [d(x), yd(z)] = 0 \tag{6}
\]

for all \( x, y, z \in I \). Replacing \( z \) by \( zd(x) \) in (6) and using (6) to get

\[
yz[d(x), d^2(x)] + [yd(x), z]d^2(x) + [d(x), y]zd^2(x) \tag{7}
\]

for all \( x, y, z \in I \). Replacing \( y \) by \( wy \) in (7) and using (7) to get

\[
[d(x), w]yzd^2(x) = 0 \tag{8}
\]

for all \( x, y, z, w \in I \).

For all \( x \in I \cap S_{a_{\sigma}}(R) \), (8) yields that \([d(x), w]yd^2(x) = 0 = [d(x), w]yI_{\sigma}(d^2(x)) \) for all \( x, y, w \in I \). Thus, we have either \([d(x), w]y = 0 \) or \( d^2(x) = 0 \) by Lemma 2.1. Suppose that \([d(x), w]y = 0 \) i.e. \([d(x), w]I = 0 \), then it is easy to see \([d(x), w] = 0 \). Consequently, for all \( x \in I \), either \([d(x), I] = 0 \) or \( d^2(x) = 0 \). Note that the arguments used in the proof of Theorem 3.1 are still valid in the present situation, as required. \( \square \)

**Theorem 3.3** Let \( R \) be a 2-torsion free \( \sigma \)-prime ring with an involution \( \sigma \), \( I \) a nonzero \( \sigma \)-ideal. If \( R \) admits a generalized derivation \( F \) associated a nonzero derivation \( d \) commuting with \( \sigma \) such that \( d(x) \circ F(y) = x \circ y \) for all \( x, y \in I \), then \( R \) is commutative.
Proof: If \( F = 0 \), then \( x \circ y = 0 \) for all \( x, y \in I \). Replacing \( y \) by \( yz \) and using that \( x \circ y = 0 \) to get \( y[x, z] = 0 \) for all \( x, y, z \in I \). In particular, \([x, z]I[x, z] = 0 = [x, z]I\sigma([x, z])\), then \([x, z] = 0\) in view of Lemma 2.1. From ([8], proof of Theorem 1.1) this yields that \( R \) is commutative.

If \( F \neq 0 \), then \( d(x) \circ F(y) = x \circ y \) for all \( x, y \in I \). Replacing \( y \) by \( yr \) to get

\[
(d(x) \circ y)d(r) - y[d(x), d(r)] + (d(x) \circ F(y))r - F(y)[d(x), r] = (x \circ y)r - y[x, r]
\]

which reduces to

\[
(d(x) \circ y)d(r) - y[d(x), d(r)] - F(y)[d(x), r] + y[x, r] = 0\quad (9)
\]

for all \( x, y \in I \) and \( r \in R \). In (9), replacing \( r \) by \( d(x) \) to get

\[
(d(x) \circ y)d^2(x) - y[d(x), d^2(x)] + y[x, d(x)] = 0\quad (13)
\]

for all \( x, y \in I \). Replacing \( y \) by \( zy \) in (10) and using (10) to get

\[
[d(x), z]yd^2(x) = 0\quad (11)
\]

for all \( x, y, z \in I \). Now again use the arguments used in the proof of Theorem 3.1, we get the required result. \( \square \)

Theorem 3.4 Let \( R \) be a 2-torsion free \( \sigma \)-prime ring with an involution \( \sigma \), \( I \) a nonzero \( \sigma \)-ideal. If \( R \) admits a generalized derivation \( F \) associated a nonzero derivation \( d \) commuting with \( \sigma \) such that \( d(x)F(y) - xy \in Z(R) \) for all \( x, y \in I \), then \( R \) is commutative.

Proof: If \( F = 0 \), then \( xy \in Z(R) \) for all \( x, y \in I \). In particular, \([xy, z] = 0\) and hence \( x[y, z] + [x, z]y = 0 \) for all \( x, y, z \in I \). Replacing \( x \) by \( wz \) to get \([w, z]xy = 0\) for all \( w, x, y, z \in I \) and therefore \([w, z]Iy = 0 = [w, z]I\sigma(y)\). Applying Lemma 2.1, we get \([w, z] = 0\) for all \( w, z \in I \) and from ([8], proof of Theorem 1.1) we get the required result. \( \square \)

If \( F \neq 0 \), then \( d(x)F(y) - xy \in Z(R) \) for all \( x, y \in I \). Replacing \( y \) by \( yz \) to get \((d(x)F(y) - xy)z + d(x)yd(z) \in Z(R)\), which implies \([d(x)yd(z), z] = 0\) for all \( x, y, z \in I \). Hence it follows that \([d(x)yd(z), z] + [d(x), z]yd(z) = 0\) for all \( x, y, z \in I \). Replacing \( y \) by \( d(x)y \) in the above to get

\[
[d(x), z]d(x)yd(z) = 0\quad (12)
\]

for all \( x, y, z \in I \). For all \( z \in I \cap Sa_\sigma(R) \), (12) yields that \([d(x), z]d(x) = 0\) or \( d(z) = 0\) by Lemma 2.1. For any \( z \in I \), the fact \( z - \sigma(z) \in I \cap Sa_\sigma(R) \) yields that either \( d(z - \sigma(z)) = 0 \) or \([d(x), z - \sigma(z)]d(x) = 0\). If \( d(z - \sigma(z)) = 0 \), then \( d(z) = \sigma(d(z)) \) and hence (12) yields that \([d(x), z]d(x) = 0\) or \( d(z) = 0\). If \([d(x), z - \sigma(z)]d(x) = 0\), using that \( z + \sigma(z) \in I \cap Sa_\sigma(R) \) then \([d(x), z + \sigma(z)]d(x) = 0\) or \( d(z + \sigma(z)) = 0\). Assume that \([d(x), z + \sigma(z)]d(x) = 0\), then \( 2[d(x), z]d(x) = 0\).
and hence \([d(x), z]d(x) = 0\). Assume that \(d(z + \sigma(z)) = 0\), then \(d(z) = -\sigma(d(z))\) and hence (12) yields that \([d(x), z]d(x) = 0\) or \(d(z) = 0\). Consequently, for all \(z \in I\), either \([d(x), z]d(x) = 0\) or \(d(z) = 0\).

Now let \(I_1 = \{ z \in I \mid [d(x), z]d(x) = 0 \}\) and \(I_2 = \{ z \in I \mid d(z) = 0 \}\). Then \(I_1, I_2\) are both additive subgroups of \(I\) and \(I_1 \cup I_2 = I\). By Brauer’s trick, either \(I_1 = I\) or \(I_2 = I\).

On the one hand, if \(I_1 = I\) then \([d(x), z]d(x) = 0\), and hence \([d(x), yz]d(x) = 0\), from ([5], proof of Theorem 2.1) \(R\) is commutative.

On the other hand, if \(I_2 = I\) then \(d(I) = 0\) and \(R\) is commutative by Lemma 2.3.

The following example demonstrates that the above results are not true in the case of arbitrary rings.

**Example 3.1.** Let \(Z\) be the ring of integers. Set \(R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Z \right\}\) and \(I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in Z \right\}\). We define the following maps: \(\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}\), \(F \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 2b \\ 0 & 0 \end{pmatrix}\), \(d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\). Then it is easy to see that \(I\) is a \(\sigma\)-ideal of \(R\) with an involution \(\sigma\) and \(F\) is a generalized derivation associated with a nonzero derivation \(d\) commuting with \(\sigma\). Moreover, it is straightforward to check that \(F\) satisfies the properties: (i) \(d(x) \circ F(y) = 0\) (ii) \([d(x), F(y)] = 0\) (iii) \(d(x) \circ F(y) = x \circ y\) (iv) \(d(x)F(y) - xy \in Z(R)\) for all \(x, y \in I\). However, \(R\) is not commutative.

**Remark 3.1.** Some more concrete examples showing the hypothesis of \(\sigma\)-primeness is necessary for \(R\) in literature appear in the works of Oukhtite [14], [15] and [16].

4. \((\alpha, \beta)\)-derivations of \(\sigma\)-prime rings

**Theorem 4.1** Let \(R\) be a 2-torsion free \(\sigma\)-prime ring with an involution \(\sigma\), \(I\) a nonzero \(\sigma\)-ideal and \(G\) an \((\alpha, \beta)\)-derivation commuting with \(\sigma\), where \(\beta\) is a automorphism of \(R\) such that \(\sigma \beta = \beta \sigma\). If \(G\) acts as an homomorphism or as an anti-homomorphism on \(I\), then \(G = 0\) or \(G = \beta\) on \(I\).

**Proof:** Assume that \(G\) acts as a homomorphism on \(I\). By our hypothesis, we have \(G(xy) = G(x)G(y)\), which can be rewritten as

\[
G(x)G(y) = G(x)\alpha(y) + \beta(x)G(y)
\]

for all \(x, y \in I\).

Replacing \(x\) by \(xz\) in (13), to get

\[
G(xz)G(y) = G(xz)\alpha(y) + \beta(xz)G(y) = G(x)G(z)\alpha(y) + \beta(xz)G(y)
\]

for all \(x, y, z \in I\).
And hence
\[ G(xz)G(y) = G(x)G(z)\alpha(y) + \beta(xz)G(y) \] (14)
for all \( x, y, z \in I \). Note that \( G \) is a homomorphism on \( I \), we have also
\[ G(xz)G(y) = G(x)G(z)G(y) = G(x)G(zy) = G(x)G(z)\alpha(y) + G(x)\beta(z)G(y) \]
for all \( x, y, z \in I \). An hence
\[ G(xz)G(y) = G(x)G(z)\alpha(y) + G(x)\beta(z)G(y) \] (15)
for all \( x, y, z \in I \). Combing (14) with (15), we have \((G(x) - \beta(x))\) \(G(y) = 0 \) for all \( x, y, z \in I \), and hence \((G(x) - \beta(x))\beta(1)G(y) = 0 \). Set \( J = \beta(1) \), it is easy to see that \( J \) is a nonzero \( \sigma \)-ideal. In other words, we have
\[ ((G(x) - \beta(x))JG(y) = 0 \] (16)
Now (16) yields \((G(x) - \beta(x))J\sigma(G(y)) = 0 \) since both \( G \) commutes with \( \sigma \), and hence by Lemma 2.1 either \( G(x) - \beta(x) = 0 \) or \( G(y) = 0 \) for all \( x, y \in I \), namely, \( G = \beta \) or \( G = 0 \) on \( I \).

Now assume that \( G \) acts as an anti-homomorphism on \( I \), then \( G(xy) = G(y)G(x) \), which can be rewritten as
\[ G(y)G(x) = G(x)\alpha(y) + \beta(x)G(y) \] (17)
for all \( x, y \in I \). Replacing \( x \) by \( xy \) in (17) to get \( G(y)G(xy) = G(xy)\alpha(y) + \beta(xy)G(y) \), which implies that \( G(y)G(x)\alpha(y) + G(y)\beta(x)G(y) = G(y)G(x)\alpha(y) + \beta(xy)G(y) \), hence we have
\[ G(y)\beta(x)G(y) = \beta(xy)G(y) \] (18)
for all \( x, y \in I \). Replacing \( x \) by \( rx \) in (18) and using (18) to get
\[ [G(y), \beta(r)]\beta(x)G(y) = 0 \] (19)
or equivalently, if we set \( J = \beta(1) \) then we have
\[ [G(y), \beta(r)]J\sigma(G(y) = 0 \] (20)
for all \( y \in I \) and \( r \in R \). For all \( y \in I \cap S_{\alpha}(R) \), we have \([G(y), \beta(r)]J\sigma(G(y) = 0 = [G(y), \beta(r)]J\sigma(G(y)) \) from (20), and hence \([G(y), \beta(r)] = 0 \) or \( G(y) = 0 \) by Lemma 2.1. But \( G(y) = 0 \) also implies that \([G(y), \beta(r)] = 0 \). Accordingly, for all \( y \in I \cap S_{\alpha}(R) \) we have \([G(y), \beta(r)] = 0 \) for all \( r \in R \). For all \( y \in I \), as \( y - \sigma(y) \in I \cap S_{\alpha}(R) \) yields that \([G(y - \sigma(y)), \beta(r)] = 0 \). Therefore \([G(y), \beta(r)] = [G(\sigma(y)), \beta(r)] \) and the relation (24) gives us \([G(y), \beta(r)]JG(y) = 0 = [G(\sigma(y)), \beta(r)]JG(y) = \sigma([G(\sigma(y)), \beta(r)]JG(y). \) Using Lemma 2.1, we get \([G(y), \beta(r)] = 0 \) or \( G(y) = 0 \), in which case \([G(y), \beta(r)] = 0 \). Consequently, for all \( y \in I \) we have \([G(y), \beta(r)] = 0 \) i.e., \([G(y), R] = 0 \) and then \( G(I) \subseteq Z(R) \). Hence \( G \) acts as a homomorphism on \( I \) so that \( G = 0 \) or \( G = \beta \) on \( I \). \( \square \)
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